MINIMAL CLASS THEOREMS IN MEASURE THEORY

By

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TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate Council
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We formulate and prove two general results on minimal classes which are then used to establish several minimal class theorems in measure theory. None of the proofs, including those which are used to extend the functional version of the monotone class theorem and its variants, require Zorn's Lemma. In addition to new monotone class theorems, we are also able to derive several variants and extensions of existing ones. The generality of the approach allows most of the variants to be easily stated in terms of more than two classes or sets. In particular, we show that the usual monotone class theorem and the π-λ theorem are corollaries to one such variant. We obtain, in addition, a theorem for σ-rings corresponding to the π-λ theorem. The extensions are mostly confined to replacement of closure conditions on the smaller class by less restrictive requirements. One such extension concerns the functional

version of the π-λ theorem and is proved directly, without using the π-λ theorem. Several results on the generated ring and field are also presented, one of which enables us to extend the known result on the structure of the ring generated by a semiring.

INTRODUCTION

Results concerning minimal classes occur in many branches of mathematics. Two well known examples of minimal class theorems in measure theory are furnished by the monotone class theorem and one of its variants, the π-λ theorem (see, for example, Billingsley [1, p. 34-35, Theorems 3.2, 3.4]). The former is primarily used to prove uniqueness of an extended measure, while the latter has numerous applications in probability theory.

The present work is concerned with obtaining other minimal class theorems within the areas of measure theory and probability theory. It is perhaps of interest to note that the methods of proof we employ do not require the use of Zorn's Lemma.

Section I is devoted to the formulation and proof of two results on minimal classes that hold under very general conditions. In the remaining sections, we apply these results to derive new minimal class theorems in measure theory. Some of the consequences of the approach we have taken are the following:

(1) Functional versions of minimal class theorems for σ-fields are proved without using Zorn's Lemma. In particular, we extend the functional version of the monotone class theorem and its variants (Theorems IV.6, Corollaries

IV.11 and IV.12). Note that the proofs given in Dellacherie and Meyer [4, p. 14-15], rely on Zorn's Lemma. In this connection, we should also mention that proofs given for the usual monotone class theorem do not require Zorn's Lemma (see, however, Dellacherie and Meyer [4, p. 13, Theorem 19]).

(2) In view of Theorems I.1 and I.2, it is now relatively simple to get extensions of existing monotone class theorems, without examining the proofs, as is customarily done (see, for example, Dellacherie and Meyer [4. p. 15, 22.2]). These extensions consist of weakening, somewhat, the requirement that the smaller class or set be closed under certain operations.

(3) A "π-λ theorem for σ-rings" is formulated and proved (Theorem III.5).

(4) An extension of the functional version of the π-λ theorem is proved directly, without using the π-λ theorem (Theorem IV.3).

(5) Many results on the generated ring and field are obtained, one of them (Corollary II.7) being an extension of the known result on the structure of the ring generated by a semiring (see, for example, Dinculeanu [5, p. 8-9, Proposition 13]).

(6) New monotone class theorems are proved (for example, Theorems III.21 and III.24).

(7) For existing monotone class theorems, several variants involving more than two classes or sets are

derived (for example, Corollary III.25 and Theorem IV.9). An interesting consequence of Corollary III.25 is that from it we immediately obtain both the monotone class theorem and the π-λ theorem.

SECTION I
PRELIMINARIES

We begin by establishing two general results on minimal classes. These will be frequently used in the sequel.

In this section, P is a set, Σ is a class of subsets of P and T is a subset of P. If V is a subset of P, then the intersection of all sets in Σ that contain V is denoted by $S(\Sigma,V)$ and the set $\{W\subset P\colon W\cap U\in\Sigma$ for some U with $V\subset U\subset P\}$ is denoted by $\Gamma(\Sigma,V)$. Whenever the notation $S(\Sigma,V)$ is used, we will assume this set to be non-empty.

<u>Theorem 1</u>. For each $b\in P$, let there be given an assertion $\alpha(b)$ concerning b. Denote by $G(\alpha)$, the set $\{b\in P\colon \alpha(b)$ is true$\}$. Suppose that

1.1 $\alpha(b)$ is true whenever $b\in T$ and

1.2 $G(\alpha)\in\Gamma$ where $\Gamma = \Gamma(\Sigma,T)$.

Then $\alpha(b)$ is true whenever $b\in S$, where $S = S(\Sigma,T)$.

<u>Proof</u>: 1.1 is equivalent to $T\subset G(\alpha)$. By 1.2, there exists U such that $T\subset U\subset P$ and $G(\alpha)\cap U\in\Sigma$. Since $T\subset G(\alpha)\cap U$, it follows by the minimality of S, that $S\subset G(\alpha)\cap U$. But then $S\subset G(\alpha)$, which is equivalent to what needed to be proved.

<u>Theorem 2</u>. Let R be a binary relation on P (that is $R\subset P\times P$) and for each subset W of P denote by G(W), the set $\{b\in P\colon aRb$

for every $a\epsilon W$ } and by H(W), the set $\{b\epsilon P: bRa$ for every $a\epsilon W\}$. Now let I be a subset of P and denote the sets $S(\Sigma, T\cap I)$, $\Gamma(\Sigma, T\cap I)$ by S, Γ respectively. Suppose that

2.1 aRb whenever $a, b\epsilon T\cap I$

2.2 $G(T\cap I)\epsilon\Gamma$ and

2.3 $H(S\cap I)\epsilon\Gamma$.

Then whenever $b\epsilon S$, $a\epsilon S\cap I$ we have bRa.

Proof: Observe that 2.1 is equivalent to $T\cap I\subset G(T\cap I)$. By 2.2, there exists U such that $T\cap I\subset U\subset P$ and $G(T\cap I)\cap U\epsilon\Sigma$. Since $T\cap I\subset G(T\cap I)\cap U$, it follows from the definition of S that $S\subset G(T\cap I)\cap U$. So $S\cap I\subset G(T\cap I)$ and thus whenever $a\epsilon T\cap I$, $b\epsilon S\cap I$ we have aRb. However, this is clearly equivalent to bRa whenever $a\epsilon S\cap I$, $b\epsilon T\cap I$ and so $T\cap I\subset H(S\cap I)$. By 2.3, there exists U' such that $T\cap I\subset U'\subset P$ and $H(S\cap I)\cap U'\epsilon\Sigma$. Since $T\cap I\subset H(S\cap I)\cap U'$, it is now clear from the definition of S that $S\subset H(S\cap I)\cap U'$. It follows that $S\subset H(S\cap I)$; so whenever $b\epsilon S$, $a\epsilon S\cap I$ we must have bRa.

Remarks. (1) Let $V\subset P$. If $W\epsilon\Sigma$, then $W\epsilon\Gamma(\Sigma,V)$ since we can take U = P in the definition of $\Gamma(\Sigma,V)$. Thus Γ may be replaced by Σ in any of the conditions 1.2, 2.2 or 2.3.

(2) Let $V\subset P$. If $W\cap S(\Sigma,V)\epsilon\Sigma$, then $W\epsilon\Gamma(\Sigma,V)$ since $V\subset S(\Sigma,V)\subset P$.

(3) We will apply Theorem 1 only in the special case when there is given a function $\theta: P\to P$ and for each $b\epsilon P$, the assertion $\alpha(b)$ is the statement: $\theta(b)\epsilon\ S(\Sigma,T)$. Then clearly,

$G(\alpha) = \{b\varepsilon P\colon \theta(b)\varepsilon S\}$, where $S = S(\Sigma,T)$, since $\alpha(b)$ is true if and only if $\theta(b)\varepsilon S(\Sigma,T)$.

(4) In Theorem 2, if R is symmetric (that is, aRb implies bRa for every $a,b\varepsilon P$), then for each subset V of P we have $G(V) = H(V)$.

(5) In all applications of Theorem 2, unless otherwise stated, we will assume that $I = P$ so that $T\cap I = T$ and $S\cap I = S$.

(6) For our purposes, the setting for Theorem 2 will always be the following: V is a subset of P and $*$ is a binary operation on P. The relation R is defined on P by aRb if and only if $a*b\varepsilon V$. Observe then that $G(W) = \{b\varepsilon P\colon a*b\varepsilon V$ for every $a\varepsilon W\}$ and $H(W) = \{b\varepsilon P\colon b*a\varepsilon V$ for every $a\varepsilon W\}$ whenever W is a subset of P. Moreover, if $*$ is commutative, then R is symmetric.

(7) By using the language of Theorem 1 we obtain the following proof of Theorem 2, which is easier to state, but nevertheless equivalent to the one already given:
For each $b\varepsilon P$, let $\alpha(b)$ be the assertion: aRb for every $a\varepsilon T\cap I$. Then 2.1 is equivalent to the statement: $\alpha(b)$ is true whenever $b\varepsilon T\cap I$. Also from 2.2 it is clear that $G(\alpha) = G(T\cap I)\varepsilon\Gamma$. Applying Theorem 1 we have $\alpha(b)$ is true whenever $b\varepsilon S$. It then follows that bRa whenever $a\varepsilon S\cap I$, $b\varepsilon T\cap I$. We thus have $\alpha'(b)$ is true whenever $b\varepsilon T\cap I$ where, for each $b\varepsilon P$, $\alpha'(b)$ is the assertion: bRa for every $a\varepsilon S\cap I$. Now $G(\alpha') = H(S\cap I)\varepsilon\Gamma$ in view of 2.3 and hence we see, by using Theorem 1 again, that $\alpha'(b)$ is true whenever $b\varepsilon S$. This completes the proof.

SECTION II
MINIMAL CLASS THEOREMS FOR RINGS AND FIELDS

In this section, we derive, by means of the results in Section I, many results concerning the generated ring and field. In particular, we are able to extend the following well known result on the structure of the ring generated by a semiring:

Let P be a semiring (that is, P is closed under finite intersections and if $A, B \in P$, then there exist disjoint sets $C_1, \ldots, C_n \in P$ such that $A-B = \cup_{i=1}^{n} C_i$). Then the ring generated by P consists precisely of all sets which are finite disjoint unions of sets in P (see, for example, Dinculeanu [5, p. 8-9, Proposition 13]). Our extension consists of weakening, somewhat, the requirement that P be a semiring.

Throughout this section, Ω is a set and $P(\Omega)$ is the class of all subsets of Ω. If T is a class of subsets of Ω, the ring and field generated by T will be denoted by $\mathcal{r}(T)$ and $\mathcal{f}(T)$ respectively.

Let V be a class of subsets of Ω. For each class W of subsets of Ω, we define the following sets:

$G_I(W) = \{B \subset \Omega : A \cap B \in V \text{ for every } A \in W\};$

$G_U(W) = \{B \subset \Omega : A \cup B \in V \text{ for every } A \in W\};$

$G_D(W) = \{B \subset \Omega:\ A-B \in V \text{ for every } A \in W\}$;

$H_D(W) = \{B \subset \Omega:\ B-A \in V \text{ for every } A \in W\}$;

$G_{SD}(W) = \{B \subset \Omega:\ A \Delta B \in V \text{ for every } A \in W\}$;

$G_C = \{B \subset \Omega:\ B^C \in V\}$.

Finally, Σ_{FI} denotes the collection of all $U \subset P(\Omega)$ such that U is closed under finite intersections;

Σ_{FU} denotes the collection of all $U \subset P(\Omega)$ such that U is closed under finite unions;

Σ_D denotes the collection of all $U \subset P(\Omega)$ such that U is closed under differences;

Σ_{PD} denotes the collection of all $U \subset P(\Omega)$ such that U is closed under proper differences;

Σ_{FDU} denotes the collection of all $U \subset P(\Omega)$ such that U is closed under finite disjoint unions;

Σ_{SD} denotes the collection of all $U \subset P(\Omega)$ such that U is closed under symmetric differences;

Σ_C denotes the collection of all $U \subset P(\Omega)$ such that U is closed under complements;

Σ_Ω denotes the collection of all $U \subset P(\Omega)$ such that $\Omega \in U$.

Theorem 1. Let $W, V \subset P(\Omega)$.

(a) If $W \subset V$, then $\Omega \in G_I(W)$; if $\Omega \in V$, then $\Omega \in G_U(W)$; if $\Phi \in V$, then $\Omega \in G_C$; if $\Phi \in V$, then $\Omega \in G_D(W)$; if $W \subset V$ and $V \in \Sigma_C$, then $\Omega \in H_D(W)$.

(b) If $V \in \Sigma_{FDU}$, then $G_I(W)$, $H_D(W) \in \Sigma_{FDU}$

(c) If $V \in \Sigma_{FI}$, then $G_D(W), G_C \in \Sigma_{FU}$ and $H_D(W), G_{SD}(W) \in \Sigma_{FI}$.

(d) If $V \in \Sigma_{FU}$, then $G_D(W)$, $G_C \in \Sigma_{FI}$ and $H_D(W) \in \Sigma_{FU}$.

(e) Let $V\varepsilon\Sigma_{PD}$. Then $G_I(W)\varepsilon\Sigma_{PD}$ and if $W\subset V$, then $G_U(W)\cap V\varepsilon\Sigma_{PD}$.

(f) Let $V\varepsilon\Sigma_{SD}$. Then $G_I(W), H_D(W)\varepsilon\Sigma_{SD}$ and if $W\subset V$, then $G_U(W)$, $G_D(W)\varepsilon\Sigma_{SD}$.

<u>Proof</u>: (a) If $W\subset V$, then $A\cap\Omega = A\varepsilon V$ for every $A\varepsilon W$ so $\Omega\varepsilon G_I(W)$. If $\Omega\varepsilon V$, then $\Omega\varepsilon G_U(W)$ since $A\cup\Omega = \Omega$ for every $A\varepsilon W$. If $\Phi\varepsilon V$, then $\Omega\varepsilon G_C$ since $\Phi = \Omega^C$. If $\Phi\varepsilon V$, then $\Omega\varepsilon G_D(W)$ since $\Phi = A-\Omega$ for every $A\varepsilon W$. Finally, suppose that $W\subset V$ and $V\varepsilon\Sigma_C$. If $A\varepsilon W$, then $A\varepsilon V$ and so $\Omega-A = A^C\varepsilon V$. Hence $\Omega\varepsilon H_D(W)$.

(b) Let $B,C\varepsilon G_I(W)$ with $B\cap C = \Phi$ and let $A\varepsilon W$. Then $A\cap B, A\cap C\varepsilon V$ and $(A\cap B)\cap(A\cap C) = \Phi$. Since $V\varepsilon\Sigma_{FDU}$ we have $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)\varepsilon V$. Thus $B\cup C\varepsilon G_I(W)$ and so $G_I(W)\varepsilon\Sigma_{FDU}$. Let $B,C\varepsilon H_D(W)$ with $B\cap C = \Phi$ and let $A\varepsilon W$. Then $B-A, C-A\varepsilon V$ and $(B\cup C)-A = (B-A)\cup(C-A)\varepsilon V$ since $V\varepsilon\Sigma_{FDU}$. Thus $B\cup C\varepsilon H_D(W)$ and so $H_D(W)\varepsilon\Sigma_{FDU}$.

(c) Let $B,C\varepsilon G_D(W)$ and let $A\varepsilon W$. Then $A-B, A-C\varepsilon V$. So $A-(B\cup C) = (A-B)\cap(A-C)\varepsilon V$ since $V\varepsilon\Sigma_{FI}$. Thus $B\cup C\varepsilon G_D(W)$ and so $G_D(W)\varepsilon\Sigma_{FU}$. Let $B,C\varepsilon G_C$. Then $B^C, C^C\varepsilon V$ and so $(B\cup C)^C = B^C\cap C^C\varepsilon V$ since $V\varepsilon\Sigma_{FI}$. It follows that $B\cup C\varepsilon G_C$ so $G_C\varepsilon\Sigma_{FU}$. Let $B,C\varepsilon H_D(W)$ and let $A\varepsilon W$. Then $B-A, C-A\varepsilon V$ and so $(B\cap C)-A = (B-A)\cap(C-A)\varepsilon V$ since $V\varepsilon\Sigma_{FI}$. This shows that $B\cap C\varepsilon H_D(W)$ so $H_D(W)\varepsilon\Sigma_{FI}$. Let $B,C\varepsilon G_{SD}(W)$ and $A\varepsilon W$. Then $A\Delta B, A\Delta C\varepsilon V$ and so $A\Delta(B\cap C) = (A\Delta B)\cap(A\Delta C)\varepsilon V$ since $V\varepsilon\Sigma_{FI}$. Thus $B\cap C\varepsilon G_{SD}(W)$ and so $G_{SD}(W)\varepsilon\Sigma_{FI}$.

(d) Let $B,C\varepsilon G_D(W)$ and let $A\varepsilon W$. Then $A-B, A-C\varepsilon V$ and so $A-(B\cap C) = (A-B)\cup(A-C)\varepsilon V$ since $V\varepsilon\Sigma_{FU}$. Hence $B\cap C\varepsilon G_D(W)$ and so $G_D(W)\varepsilon\Sigma_{FI}$.

Let $B,C\varepsilon G_C$. Then $B^c, C^c\varepsilon V$ and so $(B\cap C)^c = B^c\cup C^c\varepsilon V$ since $V\varepsilon\Sigma_{FU}$. Thus $B\cap C\varepsilon G_C$ and so $G_C\varepsilon\Sigma_{FI}$.

Let $B,C\varepsilon H_D(W)$ and let $A\varepsilon W$. Then $B-A, C-A\varepsilon V$ and so $(B\cup C)-A = (B-A)\cup(C-A)\varepsilon V$ since $V\varepsilon\Sigma_{FU}$. Hence $B\cup C\varepsilon H_D(W)$ and so $H_D(W)\varepsilon\Sigma_{FU}$.

(e) Let $B,C\varepsilon G_I(W)$ with $C\subset B$ and let $A\varepsilon W$. Then $A\cap B, A\cap C\varepsilon V$ and $A\cap C\subset A\cap B$.

So $A\cap(B-C) = (A\cap B)-(A\cap C)\varepsilon V$ since $V\varepsilon\Sigma_{PD}$. This shows that $B-C\varepsilon G_I(W)$ so $G_I(W)\varepsilon\Sigma_{PD}$.

Let $B,C\varepsilon G_U(W)\cap V$ with $C\subset B$ and let $A\varepsilon W$. Then $A\cup B, A\cup C\varepsilon V$ and we also have $A\cup(B-C) = (A\cup B)-[(A\cup C)-A]$. From $W\subset V$, it follows that $A\varepsilon V$. But $A\subset A\cup C$ and $(A\cup C)-A\subset A\cup B$ and thus we have $A\cup(B-C)\varepsilon V$ since $V\varepsilon\Sigma_{PD}$. Hence $B-C\varepsilon G_U(W)$. Since $V\varepsilon\Sigma_{PD}$ we see also that $B-C\varepsilon V$ and thus $B-C\varepsilon G_U(W)\cap V$. This shows that $G_U(W)\cap V\varepsilon\Sigma_{PD}$.

(f) Let $B,C\varepsilon G_I(W)$ and let $A\varepsilon W$. Then $A\cap B, A\cap C\varepsilon V$ and so $A\cap(B\Delta C) = (A\cap B)\Delta(A\cap C)\varepsilon V$ since $V\varepsilon\Sigma_{SD}$. Thus $B\Delta C\varepsilon G_I(W)$ so $G_I(W)\varepsilon\Sigma_{SD}$.

Let $B,C\varepsilon H_D(W)$ and let $A\varepsilon W$. Then $B-A, C-A\varepsilon V$ and it follows that $(B\Delta C)-A = (B-A)\Delta(C-A)\varepsilon V$ since $V\varepsilon\Sigma_{SD}$. Hence $B\Delta C\varepsilon H_D(W)$ so $H_D(W)\varepsilon\Sigma_{SD}$.

Let $B,C\varepsilon G_U(W)$ and let $A\varepsilon W$. Then $A\cup B, A\cup C\varepsilon V$. Since $W\subset V$, we have $A\varepsilon V$. In view of the identity $A\cup(B\Delta C) = [(A\cup B)\Delta(A\cup C)]\Delta A$, it now follows that $A\cup(B\Delta C)\varepsilon V$ since $V\varepsilon\Sigma_{SD}$. Thus $B\Delta C\varepsilon G_U(W)$ and so $G_U(W)\varepsilon\Sigma_{SD}$.

Let $B,C\in G_D(\mathcal{W})$ and let $A\in\mathcal{W}$. Then $A-B, A-C\in\mathcal{V}$. From $\mathcal{W}\subset\mathcal{V}$, we have $A\in\mathcal{W}$ and so the identity $A-(B\Delta C) = [(A-B)\Delta(A-C)]\Delta A$ shows that $A-(B\Delta C)\in\mathcal{V}$ since $\mathcal{V}\in\Sigma_{SD}$. Hence $B\Delta C\in G_D(\mathcal{W})$ so $G_D(\mathcal{W})\in\Sigma_{SD}$.

<u>Theorem 2</u>. Let $\mathcal{P}\subset\mathcal{D}\subset\mathcal{P}(\Omega)$. Suppose that $\mathcal{D}\in\Sigma_{PD}$ and that if $A,B\in\mathcal{P}$ we have $A\cap B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{PD},\mathcal{P})$. Then $\mathcal{S}\in\Sigma_D$.

<u>Proof</u>: Since $A-B = A-(A\cap B)$ if $A,B\subset\Omega$ and $\mathcal{S}\in\Sigma_{PD}$ from the definition of $\mathcal{S}$, it suffices to show that $\mathcal{S}\in\Sigma_{FI}$. This is proved using Theorem I.2. Let the relation R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cap B\in\mathcal{S}$. Then R is symmetric and, moreover, if $\mathcal{W}\subset\mathcal{P}(\Omega)$, taking $\mathcal{V} = \mathcal{S}$, we have $G(\mathcal{W}) = G_I(\mathcal{W})$. Since $\mathcal{S}\in\Sigma_{PD}$, from Theorem 1(e) we see that $G_I(\mathcal{P})\in\Sigma_{PD}$ and $G_I(\mathcal{S})\in\Sigma_{PD}$. Now observing that I.2.1 holds by hypothesis, we apply Theorem I.2 and it follows that $\mathcal{S}\in\Sigma_{FI}$.

<u>Remark</u>. In the above theorem we needed to assume that $A\cap B\in\mathcal{S}$ for $A,B\in\mathcal{P}$, where $\mathcal{S} = S(\Sigma_{PD},\mathcal{P})$. This is somewhat more general than the requirement $\mathcal{P}\in\Sigma_{FI}$. In fact, if $\mathcal{P}\in\Sigma_{FI}$ then $A\cap B\in\mathcal{S}$ for $A,B\in\mathcal{P}$ since $\mathcal{P}\subset\mathcal{S}$. Note also that if for $A,B\in\mathcal{P}$ we assume there exist $C,D\in\mathcal{P}$ with $D\subset C$ and $A\cap B = C-D$, then $A\cap B\in\mathcal{S}$.

<u>Theorem 3</u>. Let $\mathcal{P}\subset\mathcal{U}\subset\mathcal{P}(\Omega)$ with $\mathcal{U}\in\Sigma_{FDU}$. Suppose that for $A,B\in\mathcal{P}$ we have $A\cap B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{FDU},\mathcal{P})$. Then $\mathcal{S}\in\Sigma_{FI}$.

<u>Proof</u>: Apply Theorem I.2. Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cap B\in\mathcal{S}$. Then R is symmetric. With $\mathcal{V} = \mathcal{S}$, we

note that $G(\mathcal{W}) = G_I(\mathcal{W})$ for $\mathcal{W} \subset \mathcal{P}(\Omega)$. The definition of $\mathcal{S}$ shows that $\mathcal{S} \in \Sigma_{FDU}$ and from Theorem 1(b) it follows that $G_I(\mathcal{P}) \in \Sigma_{FDU}$ and $G_I(\mathcal{S}) \in \Sigma_{FDU}$. Since $A \cap B \in \mathcal{S}$ for $A, B \in \mathcal{P}$ by hypothesis, we may apply Theorem I.2 to obtain $\mathcal{S} \in \Sigma_{FI}$.

Theorem 4. Let $\mathcal{D} \subset \mathcal{U} \subset \mathcal{P}(\Omega)$ with $\mathcal{U} \in \Sigma_{FDU}$. Suppose that for $A, B \in \mathcal{D}$ we have $A \cap B \in \mathcal{S}$ and $A-B \in \mathcal{S}'$, where $\mathcal{S} = S(\Sigma_{FDU}, \mathcal{D})$ and $\mathcal{S}' = S(\Sigma_{FDU} \cap \Sigma_{FI}, \mathcal{D})$. Then $\mathcal{r}(\mathcal{D}) \subset \mathcal{U}$.

Proof: Define R on $\mathcal{P}(\Omega)$ by ARB if and only if $A-B \in \mathcal{S}$. From Theorem 1(b,c), with $\mathcal{V} = \mathcal{S}$, we have $H_D(\mathcal{S})$, $G_D(\mathcal{D}) \in \Sigma_{FDU}$ since $\mathcal{S} \in \Sigma_{FDU} \cap \Sigma_{FI}$ by Theorem 3. Again by Theorem 3, $\mathcal{S} = \mathcal{S}'$ and so I.2.1 holds. By Theorem I.2, $\mathcal{S} \in \Sigma_D$, so $\mathcal{S}$ is a ring and the result follows since $\mathcal{D} \subset \mathcal{S} \subset \mathcal{U}$.

Remark. As with Theorem 2, we note the following. Let $A, B \in \mathcal{D}$. Then if $\mathcal{D} \in \Sigma_{FI}$, we have $A \cap B \in \mathcal{S}$ and if $\mathcal{D} \in \Sigma_D$, then $A-B \in \mathcal{S}'$. In order that $A-B \in \mathcal{S}'$, we may either assume that there exist disjoint sets $C_1, \ldots, C_n$ with $A-B = \cup_{i=1}^{n} C_i$ or else that $A-B = C \cap D$ for some $C, D \in \mathcal{D}$. The point of this remark and the one following Theorem 2, is that similar observations can be made with regard to the hypotheses of all theorems of this kind that we will be proving.

Corollary 5. Let $\mathcal{P} \subset \mathcal{D} \subset \mathcal{U}$ with $\mathcal{U} \in \Sigma_{FDU}$ and $\mathcal{D} \in \Sigma_{PD}$. Suppose that $A \cap B \in \mathcal{S}$ whenever $A, B \in \mathcal{P}$, where $\mathcal{S} = S(\Sigma_{PD}, \mathcal{P})$. Then $\mathcal{r}(\mathcal{P}) \subset \mathcal{U}$.

<u>Proof</u>: By Theorem 2, $S\varepsilon\Sigma_D$. So if A,BεS , it is clear that $A\cap B = A-(A-B)\varepsilon S$ and thus $A-B \varepsilon S(\Sigma_{FDU}\cap\Sigma_{FI},S)$ and $A\cap B \varepsilon S(\Sigma_{FDU},S)$. Since $S\subset U$ we can now apply Theorem 4, which yields $\pi(S)\subset U$. But $P\subset S$ and so the result is proved.

<u>Theorem 6</u>. Let $P\subset U\subset P(\Omega)$ with $U\varepsilon\Sigma_{FDU}\cap\Sigma_{PD}$. Suppose that $A\cap B\varepsilon S$ whenever A,BεP, where $S = S(\Sigma_{FDU}\cap\Sigma_{PD},P)$. Then $\pi(P)\subset U$.

<u>Proof</u>: Since $P\subset S\subset U$, it is clear that the result is immediate if we show that S is a ring. This, in turn, will follow once we prove that $S\varepsilon\Sigma_{FI}$. To this end, let R be defined on $P(\Omega)$ by ARB if and only if $A\cap B\varepsilon S$. Note that R is symmetric. Since $S\varepsilon\Sigma_{FDU}\cap\Sigma_{PD}$, which is easily shown to be a consequence of the definition of S, we take $V = S$ and apply Theorem 1(b,e). Then, $G_I(P), G_I(S)\varepsilon\Sigma_{FDU}\cap\Sigma_{PD}$. Now I.2.1 holds by hypothesis and so from Theorem I.2 we conclude that $S\varepsilon\Sigma_{FI}$.

<u>Remark</u>. Note that $S(\Sigma_{PD},P)\subset S(\Sigma_{FDU}\cap\Sigma_{PD},P)$ and equality does not hold in general. As such, it is not possible to derive Theorem 6 from Corollary 5. On the other hand, Corollary 5 does not follow from Theorem 6 either, although these results seem to be related. It is this kind of situation that leads to the presence of many different versions of a single minimal class theorem. Each version requires its own proof since it is usually not a corollary to the theorem or the other versions.

In the next result, we determine the structure of the ring generated by a certain class and in particular by a semiring.

Corollary 7. Let $\mathcal{D}\subset\mathcal{P}(\Omega)$ and suppose that if $A,B\in\mathcal{D}$ then $A\cap B\in\mathcal{S}$ and $A-B\in\mathcal{S}'$, where $\mathcal{S} = S(\Sigma_{FDU},\mathcal{D})$ and $\mathcal{S}' = S(\Sigma_{FDU}\cap\Sigma_{FI},\mathcal{D})$. Then $\mathcal{R}(\mathcal{D}) = \mathcal{U}$ where $\mathcal{V}$ is the class of all finite disjoint unions of sets in $\mathcal{D}$.

Proof: It is easy to see that $\mathcal{U}\varepsilon\Sigma_{FDU}$. Therefore, Theorem 4 applies and we have $\mathcal{R}(\mathcal{D})\subset\mathcal{U}$. Conversely, any set in $\mathcal{U}$ is a finite disjoint union of sets in $\mathcal{D}$ and hence of sets in $\mathcal{R}(\mathcal{D})$. Since $\mathcal{R}(\mathcal{D})$ is a ring, it follows that $\mathcal{U}\subset\mathcal{R}(\mathcal{D})$.

Theorem 8. Let $\mathcal{D}\subset\mathcal{U}\subset\mathcal{P}(\Omega)$ with $\mathcal{U}\varepsilon\Sigma_{FI}\cap\Sigma_{FU}$. Suppose that $A-B\varepsilon\mathcal{S}$ for $A,B\varepsilon\mathcal{D}$, where $\mathcal{S} = S(\Sigma_{FI}\cap\Sigma_{FU},\mathcal{D})$. Then $\mathcal{R}(\mathcal{D})\subset\mathcal{U}$.

Proof: It clearly suffices to show that $\mathcal{S}\varepsilon\Sigma_D$, for then $\mathcal{S}$ is a ring and the result follows. Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A-B\varepsilon\mathcal{S}$. Since $\mathcal{S}\varepsilon\Sigma_{FI}\cap\Sigma_{FU}$, we see from Theorem 1(c,d), with $\mathcal{V} = \mathcal{S}$, that $G_D(\mathcal{D})$, $H_D(\mathcal{S})$ $\varepsilon\Sigma_{FI}\cap\Sigma_{FU}$. Now I.2.1 is satisfied by hypothesis and so Theorem I.2 applies. Thus $\mathcal{S}\varepsilon\Sigma_D$ as desired.

Theorem 9. Let $\mathcal{U}\subset\mathcal{D}\subset\mathcal{P}(\Omega)$ with $\mathcal{D}\varepsilon\Sigma_{PD}$. Suppose that if $A,B\varepsilon\mathcal{U}$, we have $A\cup B\varepsilon\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{PD},\mathcal{U})$. Then $\mathcal{R}(\mathcal{U})\subset\mathcal{D}$.

Proof: Again we apply Theorem I.2. Define R on $\mathcal{P}(\Omega)$ by ARB if and only if $A \cup B \in \mathcal{S}$. Then R is symmetric and taking $\mathcal{V} = \mathcal{S}$, we see from Theorem 1(e) that $G_U(\mathcal{U}) \cap \mathcal{S}, G_U(\mathcal{S}) \cap \mathcal{S} \in \Sigma_{PD}$. Moreover, I.2.1 holds by hypothesis and from Theorem I.2 we have $\mathcal{S} \in \Sigma_U$. If $A, B \in \mathcal{S}$, then in view of the identity $A-B = (A \cup B)-B$, it follows that $A-B \in \mathcal{S}$ and hence $\mathcal{S}$ is a ring. Since $\mathcal{U} \subset \mathcal{S} \subset \mathcal{D}$, the proof is complete.

Theorem 10. Let $\mathcal{P} \subset \mathcal{E} \subset \mathcal{P}(\Omega)$ with $\mathcal{E} \in \Sigma_{SD}$. Suppose that $A \cap B \in \mathcal{S}$ whenever $A, B \in \mathcal{P}$, where $\mathcal{S} = S(\Sigma_{SD}, \mathcal{P})$. Then $\mathcal{R}(\mathcal{P}) \subset \mathcal{E}$.

Proof: Define R on $\mathcal{P}(\Omega)$ by ARB if and only if $A \cap B \in \mathcal{P}$. Since $\mathcal{S} \in \Sigma_{SD}$, we take $\mathcal{V} = \mathcal{S}$ and apply Theorem 1(f) to get $G_I(\mathcal{P})$, $G_I(\mathcal{S}) \in \Sigma_{SD}$. According to the hypothesis, I.2.1 holds and so by Theorem I.2 we have $\mathcal{S} \in \Sigma_{FI}$. Once we observe that $\Sigma_{SD} \subset \Sigma_{PD} \cap \Sigma_{FDU}$, it follows that $\mathcal{S}$ is a ring and the proof is complete.

Theorem 11. Let $\mathcal{U} \subset \mathcal{E} \subset \mathcal{P}(\Omega)$ with $\mathcal{E} \in \Sigma_{SD}$. Suppose that for $A, B \in \mathcal{U}$ we have $A \cup B \in \mathcal{S}$, where $\mathcal{S} = S(\Sigma_{SD}, \mathcal{U})$. Then $\mathcal{R}(\mathcal{U}) \subset \mathcal{E}$.

Proof: We follow the proofs of the previous results and apply Theorem I.2. Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A \cup B \in \mathcal{S}$. Then R is symmetric and I.2.1 holds. Also $\mathcal{S} \in \Sigma_D$ and thus, taking $\mathcal{V} = \mathcal{S}$, we note that $G_U(\mathcal{U})$, $G_U(\mathcal{S}) \in \Sigma_{SD}$ from Theorem 1(f). From Theorem I.2, $\mathcal{S} \in \Sigma_U$ and so the identity $A-B = (A \cup B)-B$ for $A, B \subset \Omega$, together with the fact

that $\Sigma_{SD} \subset \Sigma_{PD}$ shows $S \varepsilon \Sigma_D$. Hence S is a ring and the result is proved.

<u>Theorem 12</u>. Let $\mathcal{D} \subset \mathcal{E} \subset P(\Omega)$ with $\mathcal{E} \varepsilon \Sigma_{SD}$. Suppose that for $A, B \varepsilon \mathcal{D}$ we have $A-B \varepsilon S$, where $S = S(\Sigma_{SD}, \mathcal{D})$. Then $r(\mathcal{D}) \subset \mathcal{E}$.

<u>Proof</u>: We define R on $P(\Omega)$ by ARB if and only if $A-B \varepsilon S$. If we take $V = S$, then since $S \varepsilon \Sigma_{SD}$ we can apply Theorem 1(f) to get $G_D(\mathcal{D})$, $H_D(S) \varepsilon \Sigma_{SD}$. Since I.2.1 holds, we see that $S \varepsilon \Sigma_D$ from Theorem I.2. To finish the proof, we observe that S is a ring since $\Sigma_{SD} \subset \Sigma_{FDU}$.

<u>Remark</u>. Although $\Sigma_{SD} \subset \Sigma_{PD} \cap \Sigma_{FDU}$, since $S(\Sigma_{PD}, T)$, $S(\Sigma_{FDU}, T)$, $S(\Sigma_{PD} \cap \Sigma_{FDU}, T) \subset S(\Sigma_{SD}, T)$ for $T \subset P(\Omega)$ and none of the inclusions can be replaced by equality in general, it is not possible to directly obtain Theorems 10, 11 or 12 from any of our previous results.

<u>Theorem 13</u>. Let $E \subset P \subset P(\Omega)$ with $P \varepsilon \Sigma_{FI}$. Suppose that for $A, B \varepsilon E$ we have $A \Delta B \varepsilon S$, where $S = S(\Sigma_{FI}, E)$. Then $r(E) \subset P$.

<u>Proof</u>: Apply Theorem I.2 as follows. Let R be defined on $P(\Omega)$ by ARB if and only if $A \Delta B \varepsilon S$. Then R is symmetric and taking $V = S$, we note that $S \varepsilon \Sigma_{FI}$ and so $G_{SD}(E)$, $G_{SD}(S) \varepsilon \Sigma_{FI}$ in view of Theorem 1(c). It is also clear that I.2.1 is satisfied, so we apply Theorem I.2 and obtain $S \varepsilon \Sigma_{SD}$. As

before, $\Sigma_{SD} \subset \Sigma_{PD} \cap \Sigma_{FDU}$ and so it follows that S is a ring. This completes the proof.

The preceding results concerned the ring generated by a class of subsets. We now prove the corresponding theorems for the generated field.

Theorem 14. Let $C \subset U \subset P(\Omega)$ with $U \varepsilon \Sigma_{FDU}$. Suppose that for $A,B \varepsilon C$ we have $A \cap B \varepsilon S$ and $A^C \varepsilon S'$, where $S = S(\Sigma_{FDU}, C)$ and $S' = S(\Sigma_{FDU} \cap \Sigma_{FI}, C)$. Then $\mathfrak{f}(C) \subset U$.

Proof: First note that we can apply Theorem 3. It follows that $S \varepsilon \Sigma_{FI}$. It is easily verified that, in fact, $S = S'$. We now apply Theorem I.1 as follows. Define θ on $P(\Omega)$ by $\theta(A) = A^C$. Then, taking $V = S$, we see that $G_C \varepsilon \Sigma_{FDU}$ from Theorem 1(c). Since I.1.1 holds, by Theorem I.1 we have $S \varepsilon \Sigma_C$. It follows that S is a field and the result is proved.

Theorem 15. Let $P \subset D \subset P(\Omega)$ with $D \varepsilon \Sigma_{PD} \cap \Sigma_{\Omega}$. Suppose that $A \cap B \varepsilon S$ whenever $A,B \varepsilon P$, where $S = S(\Sigma_{PD} \cap \Sigma_{\Omega}, P)$. Then $\mathfrak{f}(P) \subset D$.

Proof: Let R be defined on $P(\Omega)$ by ARB if and only if $A \cap B \varepsilon S$. Taking $V = S$ and applying Theorem 1(a,e) we have $G_I(P)$, $G_I(S) \varepsilon \Sigma_{PD} \cap \Sigma_{\Omega}$ since $S \varepsilon \Sigma_{PD}$. Also I.2.1 holds by hypothesis and thus since R is symmetric, from Theorem I.2 we obtain $S \varepsilon \Sigma_{FI}$. It is now easy to verify that S is a field and the proof is finished.

<u>Theorem 16</u>. Let $\mathcal{C}\subset\mathcal{U}\subset\mathcal{P}(\Omega)$ with $\mathcal{U}\in\Sigma_{FI}\cap\Sigma_{FU}$. Suppose that $A^c\in\mathcal{S}$ whenever $A\in\mathcal{C}$, where $\mathcal{S} = S(\Sigma_{FI}\cap\Sigma_{FU},\mathcal{C})$. Then $\mathcal{f}(\mathcal{C})\subset\mathcal{U}$.

<u>Proof</u>: Define θ on $\mathcal{P}(\Omega)$ by $\theta(A) = A^c$. In order to use Theorem I.1 we first note that I.1.1 is satisfied. From Theorem 1(c,d), taking $\mathcal{V} = \mathcal{S}$, we have $G_C\in\Sigma_{FI}\cap\Sigma_{FU}$. Thus Theorem I.1 can be applied and it follows that $\mathcal{S}\in\Sigma_C$. Hence $\mathcal{S}$ is a field and the result follows.

<u>Theorem 17</u>. Let $\mathcal{U}\subset\mathcal{D}\subset\mathcal{P}(\Omega)$ with $\mathcal{D}\in\Sigma_{PD}\cap\Sigma_{\Omega}$. Suppose that for $A,B\in\mathcal{U}$ we have $A\cup B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{PD}\cap\Sigma_{\Omega},\mathcal{U})$. Then $\mathcal{f}(\mathcal{U})\subset\mathcal{D}$.

<u>Proof</u>: Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cup B\in\mathcal{S}$. Taking $\mathcal{V} = \mathcal{S}$, we see from Theorem 1(a,e) that $G_U(\mathcal{U})\cap\mathcal{S}, G_U(\mathcal{S})\cap\mathcal{S}\in\Sigma_{PD}\cap\Sigma_{\Omega}$. Now I.2.1 holds by hypothesis and so we can apply Theorem I.2. It follows that $\mathcal{S}\in\Sigma_U$ and thus $\mathcal{S}$ is a field. The result is now immediate.

<u>Theorem 18</u>. Let $\mathcal{P}\subset\mathcal{E}\subset\mathcal{P}(\Omega)$ with $\mathcal{E}\in\Sigma_{SD}\cap\Sigma_{\Omega}$. Suppose that for $A,B\in\mathcal{P}$ we have $A\cap B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{SD}\cap\Sigma_{\Omega},\mathcal{P})$. Then $\mathcal{f}(\mathcal{P})\subset\mathcal{E}$.

<u>Proof</u>: Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cap B\in\mathcal{S}$. Taking $\mathcal{V} = \mathcal{S}$, we see from Theorem 1(a,f) that $G_I(\mathcal{P}), G_I(\mathcal{S})\in\Sigma_{SD}\cap\Sigma_{\Omega}$. Since R is symmetric and I.2.1 holds, we can apply Theorem I.2 to get $\mathcal{S}\in\Sigma_{FI}$. Hence $\mathcal{S}$ is a field and the result is proved.

Theorem 19. Let $\mathcal{U}\subset\mathcal{E}\subset\mathcal{P}(\Omega)$ with $\mathcal{E}\in\Sigma_{SD}\cap\Sigma_{\Omega}$. Suppose that $A\cup B\in\mathcal{S}$ whenever $A,B\in\mathcal{U}$, where $\mathcal{S} = S(\Sigma_{SD}\cap\Sigma_{\Omega},\mathcal{U})$. Then $\mathfrak{f}(\mathcal{U})\subset\mathcal{E}$.

Proof: Apply Theorem I.2 as follows. Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cup B\in\mathcal{S}$. With $\mathcal{V} = \mathcal{S}$, from Theorem 1(a,f), we have $G_U(\mathcal{U}),G_U(\mathcal{S})\in\Sigma_{SD}\cap\Sigma_{\Omega}$. R is symmetric and I.2.1 holds by hypothesis and so we can apply Theorem I.2 which yields $\mathcal{S}\in\Sigma_U$. It is now easy to see that $\mathcal{S}$ is a field and thus the proof is complete.

Theorem 20. Let $\mathcal{D}\subset\mathcal{E}\subset\mathcal{P}(\Omega)$ with $\mathcal{E}\in\Sigma_{SD}\cap\Sigma_{\Omega}$. Suppose that whenever $A,B\in\mathcal{D}$ we have $A-B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{SD}\cap\Sigma_{\Omega},\mathcal{D})$. Then $\mathfrak{f}(\mathcal{D})\subset\mathcal{E}$.

Proof: Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A-B\in\mathcal{S}$. Then R is symmetric. Next note that $\mathcal{S}\in\Sigma_C$ since $\Sigma_{SD}\subset\Sigma_{PD}$ and $\Omega\in\mathcal{S}$. Thus we also have $\Phi\in\mathcal{S}$. Now from Theorem 1(a,f), it follows that $G_D(\mathcal{D}),H_D(\mathcal{S})\in\Sigma_{SD}\cap\Sigma_{\Omega}$ on taking $\mathcal{V} = \mathcal{S}$. Since I.2.1 holds by hypothesis, we can apply Theorem I.2 and obtain $\mathcal{S}\in\Sigma_D$. Hence $\mathcal{S}$ is a field and we have the desired result.

Theorem 21. Let $\mathcal{E}\subset\mathcal{P}\subset\mathcal{P}(\Omega)$ with $\mathcal{P}\in\Sigma_{FI}$. Suppose that $A\Delta B\in\mathcal{S}$ for $A,B\in\mathcal{E}$ and $\mathcal{S}\in\Sigma_{\Omega}$, where $\mathcal{S} = S(\Sigma_{FI},\mathcal{E})$. Then $\mathfrak{f}(\mathcal{E})\subset\mathcal{P}$.

Proof: From Theorem 13, we have $\mathfrak{r}(\mathcal{E})\subset\mathcal{P}$. Now clearly $\mathcal{S}\subset\mathfrak{r}(\mathcal{E})$ and thus $\Omega\in\mathfrak{r}(\mathcal{E})$. But then $\mathfrak{r}(\mathcal{E})$ is a field and the result follows immediately.

SECTION III
MINIMAL CLASS THEOREMS FOR σ-RINGS AND σ-FIELDS

Let R be a ring and M a monotone class with $R \subset M$. (M is called a monotone class if it closed under countable increasing unions and countable decreasing intersections.) The monotone class theorem for σ-rings says that M contains the σ-ring generated by R (see Halmos [6, p. 27-28, Theorem B]). If, in addition, R is a field then M contains the σ-field generated by R (see Chung [3, p. 17-18, Theorem 2.1.2]). A similar result used in probability theory, sometimes called the π-λ theorem, and attributed to Dynkin, shows that if P is closed under finite intersections (P is called a π-system), L is closed under countable increasing unions, proper differences and Ω belongs to L (L is called a Dynkin system or λ-system) and $P \subset L$, then L contains the σ-field generated by P (see Billingsley [1, p. 34, Theorem 3.2]). Finally, a lesser known theorem on the same lines is the following: Let F be a field and let N be closed under countable disjoint unions and countable decreasing intersections (N is called a normal class) with $F \subset N$. Then N contains the σ-field generated by F (Saks [7, p. 85, 9.7]). The version of this result for σ-rings states that if D is a semiring and N a normal class with $D \subset N$, then N contains

the σ-ring generated by $\mathcal{D}$ (Halmos [6, p. 28, Exercise 3]). In this section, we employ the results in Section I to extend all of the above results, present other variants (using, in addition, results from Section II) and obtain new monotone class theorems. In particular, we prove a theorem for σ-rings corresponding to the π-λ theorem and also show that the usual monotone class theorem and π-λ theorem are, in fact, corollaries to a single variant involving more than two classes.

The following notation is used throughout this section: Ω is a set and $\mathcal{P}(\Omega)$ is the class of all subsets of Ω. The σ-ring and σ-field generated by a class $\mathcal{T}$ of subsets of Ω will be respectively denoted by $\sigma_{\mathcal{R}}(\mathcal{T})$ and $\sigma_{\mathcal{F}}(\mathcal{T})$. Moreover,

Σ_{CIU} is the collection of all $\mathcal{U}\subset\mathcal{P}(\Omega)$ such that $\mathcal{U}$ is closed under countable increasing unions;

Σ_{CDI} is the collection of all $\mathcal{U}\subset\mathcal{P}(\Omega)$ such that $\mathcal{U}$ is closed under countable decreasing intersections;

Σ_{CDU} is the collection of all $\mathcal{U}\subset\mathcal{P}(\Omega)$ such that $\mathcal{U}$ is closed under countable disjoint unions;

Σ_{CU} is the collection of all $\mathcal{U}\subset\mathcal{P}(\Omega)$ such that $\mathcal{U}$ is closed under countable unions;

Σ_{CI} is the collection of all $\mathcal{U}\subset\mathcal{P}(\Omega)$ such that $\mathcal{U}$ is closed under countable intersections.

In addition, we will use the notation of section II.

<u>Theorem 1</u>. Let $\mathcal{W}, \mathcal{V}\subset\mathcal{P}(\Omega)$.

(a) If $\mathcal{V}\in\Sigma_{CIU}$, then $\mathcal{G}_I(\mathcal{W}), \mathcal{G}_U(\mathcal{W}), \mathcal{H}_D(\mathcal{W})\in\Sigma_{CIU}$ and $\mathcal{G}_D(\mathcal{W})$, $\mathcal{G}_C\in\Sigma_{CDI}$.

(b) If $\mathcal{V}\varepsilon\Sigma_{CDI}$, then $\mathcal{G}_I(\mathcal{W}),\mathcal{G}_U(\mathcal{W}),\mathcal{H}_D(\mathcal{W})\varepsilon\Sigma_{CDI}$ and $\mathcal{G}_D(\mathcal{W})$, $\mathcal{G}_C\ \varepsilon\Sigma_{CDI}$.

(c) Let $\mathcal{V}\ \varepsilon\Sigma_{CDU}$. Then $\mathcal{G}_I(\mathcal{W}),\dot{\mathcal{H}}_D(\mathcal{W})\varepsilon\Sigma_{CDU}$; if $\mathcal{W}\subset\mathcal{V}$ and $\mathcal{V}\varepsilon\Sigma_{CI}$, then $\mathcal{G}_D(\mathcal{W})\cap\mathcal{V}\varepsilon\Sigma_{CDI}$; if $\mathcal{V}\varepsilon\Sigma_{CI}$, then $\mathcal{G}_C\cap\mathcal{V}\varepsilon\Sigma_{CDI}$.

(d) If $\mathcal{V}\ \varepsilon\Sigma_{CU}$, then $\mathcal{H}_D(\mathcal{W})\varepsilon\Sigma_{CU}$ and $\mathcal{G}_D(\mathcal{W})$, $\mathcal{G}_C\varepsilon\Sigma_{CI}$.

(e) If $\mathcal{V}\varepsilon\Sigma_{CI}$, then $\mathcal{H}_D(\mathcal{W})\varepsilon\Sigma_{CI}$ and $\mathcal{G}_D(\mathcal{W})$, $\mathcal{G}_C\varepsilon\Sigma_{CU}$.

Proof: (a) Let $B_n\uparrow B$, $B_n\varepsilon\mathcal{G}_I(\mathcal{W})$ and $A\varepsilon\mathcal{W}$. Then $A\cap B_n\varepsilon\mathcal{V}$ for each n. Since $A\cap B = A\cap(\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A\cap B_n)$ and $A\cap B_n\uparrow$, it follows that $A\cap B_n\uparrow A\cap B$. But $\mathcal{V}\varepsilon\Sigma_{CIU}$ and so $A\cap B\varepsilon\mathcal{V}$. This shows that $\mathcal{G}_I(\mathcal{W})\ \varepsilon\Sigma_{CIU}$.

Let $B_n\uparrow B, B_n\varepsilon\mathcal{G}_U(\mathcal{W})$ and $A\varepsilon\mathcal{W}$. Then $A\cup B_n\varepsilon\mathcal{V}$ for each n and since $A\cup B = A\cup(\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A\cup B_n)$ with $A\cup B_n\uparrow$, we have $A\cup B_n\uparrow A\cup B$. The assumption $\mathcal{V}\varepsilon\Sigma_{CIU}$ now shows that $A\cup B\varepsilon\mathcal{V}$. Thus $\mathcal{G}_U(\mathcal{W})\varepsilon\Sigma_{CIU}$.

Let $B_n\uparrow B, B_n\varepsilon\mathcal{H}_D(\mathcal{W})$ and $A\varepsilon\mathcal{W}$. Then $B_n-A\varepsilon\mathcal{V}$ for each n. Now $B_n-A\uparrow$ and $B-A = (\cup_{n=1}^{\infty} B_n)-A = \cup_{n=1}^{\infty}(B_n-A)$ so $B_n-A\uparrow B-A$. Since $\mathcal{V}\varepsilon\Sigma_{CIU}$, it follows that $B-A\varepsilon\mathcal{V}$ and hence $\mathcal{H}_D(\mathcal{W})\ \varepsilon\Sigma_{CIU}$.

Let $B_n\downarrow B, B_n\varepsilon\mathcal{G}_D(\mathcal{W})$ and $A\varepsilon\mathcal{W}$. Then we have $A-B_n\varepsilon\mathcal{V}$ for each n. Note that $A-B_n\uparrow A-B$ since $A-B_n\uparrow$ and $A-B = A-(\cap_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty}(A-B_n)$. But $\mathcal{V}\varepsilon\Sigma_{CIU}$ and so $A-B\varepsilon\mathcal{V}$. Thus $\mathcal{G}_D(\mathcal{W})\varepsilon\Sigma_{CDI}$.

Let $B_n\downarrow B, B_n\ \varepsilon\mathcal{G}_C$. Then $B_n{}^c\varepsilon\mathcal{V}$ for each n. Since $B_n{}^c\uparrow$ and $B^c = (\cap_{n=1}^{\infty} B_n)^c = \cup_{n=1}^{\infty} B_n{}^c$, we have $B_n{}^c\uparrow B^c$. Now $\mathcal{V}\varepsilon\Sigma_{CIU}$ and thus $B^c\varepsilon\mathcal{V}$, so it follows that $\mathcal{G}_C\varepsilon\Sigma_{CDI}$.

(b) Let $B_n\downarrow B, B_n\varepsilon\mathcal{G}_I(\mathcal{W})$ and $A\varepsilon\mathcal{W}$. Then $A\cap B_n\varepsilon\mathcal{V}$ for each n and since $A\cap B_n\downarrow$ and $A\cap B = A\cap(\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty}(A\cap B_n)$ we have

$A\cap B_n \downarrow A\cap B$. But we assumed $V\varepsilon\Sigma_{CDI}$ and hence $A\cap B\varepsilon V$. Therefore, it is clear that $G_I(W)\varepsilon\Sigma_{CDI}$.

Let $B_n\downarrow B, B_n\varepsilon G_U(W)$ and $A\varepsilon W$. Then $A\cup B_n\varepsilon V$ for each n. Now $A\cup B_n\downarrow$ and $A\cup B = A\cup(\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty} (A\cup B_n)$ so $A\cup B_n\downarrow A\cup B$. Thus $A\cup B\varepsilon V$ since $V\varepsilon\Sigma_{CDI}$ and hence $G_U(W)\varepsilon\Sigma_{CDI}$.

Let $B_n\downarrow B, B_n\varepsilon H_D(W)$ and $A\varepsilon W$. Then $B_n-A\varepsilon V$ for each n. Since $B-A = (\cap_{n=1}^{\infty} B_n)-A = \cap_{n=1}^{\infty} (B_n-A)$ and $B_n-A\downarrow$ we have $B_n-A\downarrow B-A$. Thus $B-A\varepsilon V$ since $V\varepsilon\Sigma_{CDI}$ and so $H_D(W)\varepsilon\Sigma_{CDI}$.

Let $B_n\uparrow B, B_n\varepsilon G_D(W)$ and $A\varepsilon W$. Then $A-B_n\varepsilon V$ for each n. Now $A-B_n\downarrow$ and $A-B = A-(\cup_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty} (A-B_n)$ so $A-B_n\downarrow A-B$. Since $V\varepsilon\Sigma_{CDI}$, we have $A-B\varepsilon V$. Hence $G_D(W)\varepsilon\Sigma_{CIU}$.

Let $B_n\uparrow B, B_n\varepsilon G_C$. Then $B_n^c\varepsilon V$ for each n. Since $B_n^c\downarrow$ and $B^c = (\cup_{n=1}^{\infty} B_n)^c = \cap_{n=1}^{\infty} B_n^c$, we get $B_n^c\downarrow B^c$. But $V\varepsilon\Sigma_{CDI}$ and so $B^c\varepsilon V$. This shows that $G_C\varepsilon\Sigma_{CIU}$.

(c) Let $B = \cup_{n=1}^{\infty} B_n$ where B_n are disjoint and $B_n\varepsilon G_I(W)$. Let $A\varepsilon W$. Then $A\cap B_n\varepsilon V$ for each n and $A\cap B_n$ are disjoint. Since we have $A\cap B = A\cap(\cup_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty} (A\cap B_n)$ and $V\varepsilon\Sigma_{CDU}$, it follows that $A\cap B\varepsilon V$. Thus $G_I(W)\varepsilon\Sigma_{CDU}$.

Let $B = \cup_{n=1}^{\infty} B_n$, where B_n are disjoint and $B_n\varepsilon H_D(W)$. Let $A\varepsilon W$. Then we have $B_n-A\varepsilon V$ for each n and B_n-A are disjoint. Now $B-A = (\cup_{n=1}^{\infty} B_n)-A = \cup_{n=1}^{\infty} (B_n-A)$ and so $B-A\varepsilon V$ since $V\varepsilon\Sigma_{CDU}$. Hence, $H_D(W)\varepsilon\Sigma_{CDU}$.

Suppose that $W\subset V$, $V\varepsilon\Sigma_{CI}$ and $A\varepsilon W$. Then $A\varepsilon V$. Now if $B_n\downarrow B$, with $B_n\varepsilon G_D(W)\cap V$, then $A-B_n\varepsilon V$ for each n. Let $C_1 = A-B_1, C_n=(A-B_n)\cap(A\cap B_{n-1})$ for $n\geq 2$. Then it follows that $C_n\varepsilon V$ for each n, since $V\varepsilon\Sigma_{FI}$. Moreover, it can be verified that C_n are disjoint since $B_n\downarrow$. Also we have $A-B = \cup_{n=1}^{\infty} C_n$. Consequently, $A-B\varepsilon V$ since $V\varepsilon\Sigma_{CDU}$ and thus

$B \in \mathcal{G}_D(\mathcal{W})$. The assumption $\mathcal{V} \in \Sigma_{CI}$ shows, in particular, that $\mathcal{V} \in \Sigma_{CDI}$ and hence $B \in \mathcal{V}$. But then it follows that $\mathcal{G}_D(\mathcal{W}) \cap \mathcal{V} \in \Sigma_{CDI}$.

Let $B_n \downarrow B, B_n \in \mathcal{G}_C \cap \mathcal{V}$. Then $B_n^c \in \mathcal{V}$ for each n. Let $C_1 = B_1^c$, $C_n = B_n^c \cap B_{n-1}$ for $n \geq 2$. Then $C_n \in \mathcal{V}$ for each n since $\mathcal{V} \in \Sigma_{FI}$ and also C_n are disjoint. Next observe that $B^c = \cup_{n=1}^{\infty} C_n$. It thus follows that $B^c \in \mathcal{V}$ since $\mathcal{V} \in \Sigma_{CDU}$. Hence $B \in \mathcal{G}_C$. Moreover, $\mathcal{V} \in \Sigma_{CDI}$ and so $B \in \mathcal{V}$. It is now clear that $\mathcal{G}_C \cap \mathcal{V} \in \Sigma_{CDI}$.

(d) Let $B = \cap_{n=1}^{\infty} B_n, B_n \in \mathcal{H}_D(\mathcal{W})$ and $A \in \mathcal{W}$. Then $B_n - A \in \mathcal{V}$ for each n. Thus $B - A = (\cap_{n=1}^{\infty} B_n) - A = \cap_{n=1}^{\infty} (B_n - A) \in \mathcal{V}$ since $\mathcal{V} \in \Sigma_{CI}$. But then $\mathcal{H}_D(\mathcal{W}) \in \Sigma_{CI}$.

Let $B = \cup_{n=1}^{\infty} B_n, B_n \in \mathcal{G}_D(\mathcal{W})$ and $A \in \mathcal{W}$. Then $A - B_n \in \mathcal{V}$ for each n. Since $\mathcal{V} \in \Sigma_{CI}$ and $A - B = A - (\cup_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty} (A - B_n)$ it follows that $A - B \in \mathcal{V}$ and so $\mathcal{G}_D(\mathcal{W}) \in \Sigma_{CU}$.

Let $B = \cup_{n=1}^{\infty} B_n, B_n \in \mathcal{G}_C$. Then $B^c = (\cup_{n=1}^{\infty} B_n)^c = \cap_{n=1}^{\infty} B_n^c$ and so $B^c \in \mathcal{V}$ since $\mathcal{V} \in \Sigma_{CI}$. Thus $\mathcal{G}_C \in \Sigma_{CU}$.

(e) Let $B = \cup_{n=1}^{\infty} B_n, B_n \in \mathcal{H}_D(\mathcal{W})$ and $A \in \mathcal{W}$. Then for each n, we have $B_n - A \in \mathcal{V}$. Since $\mathcal{V} \in \Sigma_{CU}$ and $B - A = (\cup_{n=1}^{\infty} B_n) - A = \cup_{n=1}^{\infty} (B_n - A)$, we see that $B - A \in \mathcal{V}$. This shows that $\mathcal{H}_D(\mathcal{W}) \in \Sigma_{CU}$.

Let $B = \cap_{n=1}^{\infty} B_n, B_n \in \mathcal{G}_D(\mathcal{W})$ and $A \in \mathcal{W}$. Then $A - B_n \in \mathcal{V}$ for each n. Thus $A - B = A - (\cap_{n=1}^{\infty} B_n) = \cup_{n=1}^{\infty} (A - B_n) \in \mathcal{V}$ since $\mathcal{V} \in \Sigma_{CU}$. Hence $\mathcal{G}_D(\mathcal{W}) \in \Sigma_{CI}$.

Let $B = \cap_{n=1}^{\infty} B_n, B_n \in \mathcal{G}_C$. Then $B^c = (\cap_{n=1}^{\infty} B_n)^c = \cup_{n=1}^{\infty} B_n^c \in \mathcal{V}$ since $\mathcal{V} \in \Sigma_{CU}$. So $\mathcal{G}_C \in \Sigma_{CI}$.

<u>Theorem 2</u>. Let $\mathcal{D} \subset \mathcal{U} \subset \mathcal{P}(\Omega)$ with $\mathcal{U} \in \Sigma_{CU} \cap \Sigma_{CI}$. Suppose that $A - B \in \mathcal{S}$ whenever $A, B \in \mathcal{D}$, where $\mathcal{S} = S(\Sigma_{CU}\ \Sigma_{CI}, \mathcal{D})$. Then $\sigma_{\mathcal{R}}(\mathcal{D}) \subset \mathcal{U}$.

<u>Proof</u>: Define R on $\mathcal{P}(\Omega)$ by ARB if and only if $A-B\varepsilon\mathcal{S}$. By Theorem 1(d,e), with $\mathcal{V} = \mathcal{S}$, we note that $\mathcal{G}_D(\mathcal{D})$, $\mathcal{H}_D(\mathcal{S})\varepsilon\Sigma_{CU}\cap\Sigma_{CI}$. Now I.2.1 holds by hypothesis and thus, by Theorem I.2, we have $\mathcal{S}\varepsilon\Sigma_D$. It follows that $\mathcal{S}$ is a ring and the result is proved.

<u>Remark</u>. Let $A,B\varepsilon\mathcal{D}$. As noted in Section II, if we have $\mathcal{S}\varepsilon\Sigma_D$, then $A-B\varepsilon\mathcal{S}$. Also if there exist $C_n\varepsilon\mathcal{D}$ such that either $A-B = \cup_{n=1}^{\infty} C_n$ or $A-B = \cap_{n=1}^{\infty} C_n$, then $A-B\varepsilon\mathcal{S}$ since $\mathcal{S}\varepsilon\Sigma_{CI}\cap\Sigma_{CU}$. Similar observations can be made regarding all results of this kind that are to follow.

The monotone class theorem for σ-rings is immediate from the next result.

<u>Theorem 3</u>. Let $\mathcal{D}\subset\mathcal{M}\subset\mathcal{P}(\Omega)$ with $\mathcal{M}\varepsilon\Sigma_{CIU}\cap\Sigma_{CDI}$. Suppose that whenever $A,B\varepsilon\mathcal{D}$ we have $A-B\varepsilon\mathcal{S}$ and $A\cup B\varepsilon\mathcal{S}'$ where $\mathcal{S}= S(\Sigma_{CIU}\cap\Sigma_{CDI},\mathcal{D})$ and $\mathcal{S}' = S(\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_D,\mathcal{D})$. Then $\sigma_{\mathcal{R}}(\mathcal{D})\subset\mathcal{M}$.

<u>Proof</u>: Apply Thecrem I.2 twice. First, let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A-B\varepsilon\mathcal{S}$. Now I.2.1 holds by hypothesis and with $\mathcal{V} = \mathcal{S}$, we see from Theorem 1(a,b) that $\mathcal{G}_D(\mathcal{D}),\mathcal{H}_D(\mathcal{S})\varepsilon\Sigma_{CIU}\cap\Sigma_{CDI}$. Hence by Theorem I.2 we have $\mathcal{S}\varepsilon\Sigma_D$. Next define the relation R' on $\mathcal{P}(\Omega)$ by AR'B if and only if $A\cup B\varepsilon\mathcal{S}$. Then I.2.1 holds since $\mathcal{S} = \mathcal{S}'$. Also by Theorem 1(a,b), with $\mathcal{V} = \mathcal{S}$, we have $\mathcal{G}_U(\mathcal{D}),\mathcal{G}_U(\mathcal{S})\varepsilon\Sigma_{CIU}\cap\Sigma_{CDI}$. It follows, by applying Theorem I.2 again, that $\mathcal{S}\varepsilon\Sigma_{FU}$. Thus $\mathcal{S}$ is a σ-ring and the proof is finished.

<u>Remark</u>. In the preceding theorem, an examination of the proof shows that the result is unchanged if we alter the second line of the hypothesis to read: Suppose that whenever $A,B\in\mathcal{D}$ we have $A\cup B\in\mathcal{S}$ and $A-B\in\mathcal{S}'$, where $\mathcal{S} = S(\Sigma_{CIU}\cap\Sigma_{CDI},\mathcal{D})$ and $\mathcal{S}' = S(\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{FU},\mathcal{D})$. In general, for all theorems of this kind, if we need to show closure under two different operations and if proving closure under the second operation does not require use of closure under the first, then we may "interchange" the operations in the hypothesis as demonstrated above.

<u>Theorem 4</u>. Let $\mathcal{D}\subset\mathcal{N}\subset\mathcal{P}(\Omega)$ with $\mathcal{N}\in\Sigma_{CDU}\cap\Sigma_{CDI}$. Suppose that $A\cap B\in\mathcal{S}$ and $A-B\in\mathcal{S}'$ whenever $A,B\in\mathcal{D}$, where $\mathcal{S} = S(\Sigma_{CDU}\cap\Sigma_{CDI},\mathcal{D})$, $\mathcal{S}' = S(\Sigma_{CDU}\cap\Sigma_{CI},\mathcal{D})$. Then $\sigma_{\mathcal{R}}(\mathcal{D})\subset\mathcal{N}$.

<u>Proof</u>: Again we apply Theorem I.2 twice. First, let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cap B\in\mathcal{S}$. Taking $\mathcal{V} = \mathcal{S}$, we see from Theorem 1(b,c) that $G_I(\mathcal{D}),G_I(\mathcal{S})\in\Sigma_{CDU}\cap\Sigma_{CDI}$. Since I.2.1 holds by hypothesis, we apply theorem I.2 which yields $\mathcal{S}\in\Sigma_{FI}$. Next define R' on $\mathcal{P}(\Omega)$ by AR'B if and only if $A-B\in\mathcal{S}$. Since $\mathcal{S} = \mathcal{S}'$ it follows that I.2.1 is satisfied. Take $\mathcal{V}=\mathcal{S}$. Then from Theorem 1(b,c,e) we have

$H_D(\mathcal{S}),G_D(\mathcal{D})\cap\mathcal{S}\in\Sigma_{CDU}\cap\Sigma_{CDI}$

since $\mathcal{S}\in\Sigma_{CI}\cap\Sigma_{CDU}$, $\Sigma_{CU}\subset\Sigma_{CDU}$ and Σ_{CDU} is closed under finite intersections. Thus applying Theorem I.2 we obtain $\mathcal{S}\in\Sigma_D$, so $\mathcal{S}$ is a σ-ring and the result follows.

<u>Remark</u>. The "σ-ring" version of the result on normal

classes, referred to at the beginning of this section, trivially follows from the preceding result.

We present now the "π-λ theorem for σ-rings".

Theorem 5. Let $\mathcal{P}\subset\mathcal{M}\subset\mathcal{P}(\Omega)$ with $\mathcal{M}\in\Sigma_{CIU}\cap\Sigma_{FDU}\cap\Sigma_{PD}$. Suppose that whenever $A,B\in\mathcal{P}$ we have $A\cap B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{CIU}\cap\Sigma_{FDU}\cap\Sigma_{PD},\mathcal{P})$. Then $\sigma_{\mathcal{R}}(\mathcal{P})\subset\mathcal{M}$.

Proof: Define R on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cap B\in\mathcal{S}$. From Theorem 1(a) and Theorem II.1(b,e), on taking $\mathcal{V}=\mathcal{S}$, we have $\mathcal{G}_I(\mathcal{P}),\mathcal{G}_I(\mathcal{S})\in\Sigma_{CIU}\cap\Sigma_{FDU}\cap\Sigma_{PD}$. Now I.2.1 holds by hypothesis and so by Theorem I.2, $\mathcal{S}\in\Sigma_{FI}$. It is now easily shown that $\mathcal{S}$ is a σ-ring and the proof is finished.

Remarks (1) In view of the identity $\cap_{n=1}^{\infty} A_n = A_1-[\cup_{n=2}^{\infty}(A_1-A_n)]$ for $A_n\downarrow$, we have $\Sigma_{CIU}\cap\Sigma_{PD}\subset\Sigma_{CDI}$. This fact is needed in the proof.

(2) It is easy to verify that $\Sigma_{CIU}\cap\Sigma_{FDU}\cap\Sigma_{PD} = \Sigma_{CDU}\cap\Sigma_{PD}$. Thus an equivalent form of the above result is obtained by replacing $\Sigma_{CIU}\cap\Sigma_{FDU}\cap\Sigma_{PD}$ by $\Sigma_{CDU}\cap\Sigma_{PD}$ and $S(\Sigma_{CIU}\cap\Sigma_{FDU}\cap\Sigma_{PD},\mathcal{P})$ by $S(\Sigma_{CDU}\cap\Sigma_{PD},\mathcal{P})$.

Theorem 6. Let $\mathcal{U}\subset\mathcal{M}\subset\mathcal{P}(\Omega)$ with $\mathcal{M}\in\Sigma_{CIU}\cap\Sigma_{PD}$. Suppose that for $A,B\in\mathcal{U}$ we have $A\cup B\in\mathcal{S}$, where $\mathcal{S} = S(\Sigma_{CIU}\cap\Sigma_{PD},\mathcal{U})$. Then $\sigma_{\mathcal{R}}(\mathcal{U})\subset\mathcal{M}$.

Proof: Apply Theorem I.2 as follows. Let R be defined on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cup B\in\mathcal{S}$. Taking $\mathcal{V} = \mathcal{S}$, from Theorem

1(a) and Theorem II.1(e), we have

$G_U(U)\cap S,\ G_U(S)\cap S\in\Sigma_{CIU}\cap\Sigma_{PD}$ since Σ_{CIU} is closed under finite intersections. Now I.2.1. holds by hypothesis and so by Theorem I.2, $S\in\Sigma_{FU}$. Thus S is a σ-ring and the result follows.

Theorem 7. Let $T\subset M\subset P(\Omega)$ with $M\in\Sigma_{CIU}\cap\Sigma_{SD}$. Let S = $S(\Sigma_{CIU}\cap\Sigma_{SD},T)$ and suppose that one of the following holds: Either $A\cap B\in S$ whenever $A,B\in T$, or $A\cup B\in S$ whenever $A,B\in T$, or $A-B\in S$ whenever $A,B\in T$. Then $\sigma_{\mathcal{R}}(T)\subset M$.

Proof: Let R be defined on $P(\Omega)$ by ARB if and only if $A*B\in S$ where * is either $\cap$, $\cup$ or $-$, as is appropriate. Then I.2.1 holds. Taking $V = S$, if * is $\cap$ or $\cup$, we apply Theorem 1(a) and Theorem II.1(f) to get $G_I(T),G_I(S)$ or $G_U(T),G_U(S)\in\Sigma_{CIU}\cap\Sigma_{SD}$, and if * = $-$ we apply Theorem 1(a,b) and Theorem II.1(f) to get $G_D(T),H_D(S)\in\Sigma_{CIU}\cap\Sigma_{SD}$. By Theorem I.2, we have $S\in\Sigma_{FI}$ or $S\in\Sigma_{FU}$ or $S\in\Sigma_D$. In either case, it is easy to show that S is a σ-ring and this completes the proof.

Remarks. (1) In the above proof it was assumed that $\Sigma_{CIU}\cap\Sigma_{SD}\subset\Sigma_{CDI}$, a fact which is easily verified.

(2) Since $\Sigma_{CIU}\cap\Sigma_{SD} = \Sigma_{CDU}\cap\Sigma_{SD}$, we can replace $\Sigma_{CIU}\cap\Sigma_{SD}$ by $\Sigma_{CDU}\cap\Sigma_{SD}$ and $S(\Sigma_{CIU}\cap\Sigma_{SD},T)$ by $S(\Sigma_{CDU}\cap\Sigma_{SD},T)$ to obtain an equivalent form of the preceding result.

Theorem 8. Let $\mathcal{D}\subset\mathcal{M}\subset\mathcal{P}(\Omega)$ with $\mathcal{M}\in\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{FDU}$. Suppose that $A\cap B\in\mathcal{S}$ and $A-B\in\mathcal{S}'$ whenever $A,B\in\mathcal{D}$, where $\mathcal{S} = S(\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{FDU},\mathcal{D})$ and $\dot{\mathcal{S}}' = S(\Sigma_{CIU}\cap\Sigma_{CI}\cap\Sigma_{FDU},\mathcal{D})$. Then $\sigma_{\mathcal{R}}(\mathcal{D})\subset\mathcal{M}$.

Proof: Apply Theorem I.2 twice. First, define R on $\mathcal{P}(\Omega)$ by ARB if and only if $A\cap B\in\mathcal{S}$. With $\mathcal{V} = \mathcal{S}$, we apply Theorem 1(a,b) and Theorem II.1(b). Then we have $\mathcal{G}_I(\mathcal{D}),\mathcal{G}_I(\mathcal{S})\in\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{FDU}$. Now I.2.1 holds by hypothesis and so, from Theorem I.2, we obtain $\mathcal{S}\in\Sigma_{FI}$. Next, define R' on $\mathcal{P}(\Omega)$ by AR'B if and only if $A-B\in\mathcal{S}$. Then $\mathcal{S}=\mathcal{S}'$ and so I.2.1 holds. Taking $\mathcal{V} = \mathcal{S}$, we see from Theorem 1(a,b) and Theorem II.1(b,c) that $\mathcal{G}_D(\mathcal{D}),\mathcal{H}_D(\mathcal{S})\in\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{FDU}$ since $\mathcal{S}\in\Sigma_{CIU}\cap\Sigma_{CI}\cap\Sigma_{FDU}$. Thus on applying Theorem I.2, we have $\mathcal{S}\in\Sigma_D$ and so $\mathcal{S}$ is a σ-ring. This establishes the result.

Remark: Theorem 8 does not follow from Theorem 4. Likewise, no part of Theorem 7 is a direct consequence of any of the theorems preceding it.

The next few results are corollaries to some of the results already proved.

Corollary 9. Let $\mathcal{D}\subset\mathcal{U}\subset\mathcal{M}\subset\mathcal{P}(\Omega)$ with $\mathcal{M}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ and $\mathcal{U}\in\Sigma_{FDU}$. Suppose that $A\cap B\in\mathcal{S}$ and $A-B\in\mathcal{S}'$ whenever $A,B\in\mathcal{D}$, where $\mathcal{S} = S(\Sigma_{FDU},\mathcal{D})$, $\mathcal{S}'= S(\Sigma_{FDU}\cap\Sigma_{FI},\mathcal{D})$. Then $\sigma_{\mathcal{R}}(\mathcal{D})\subset\mathcal{M}$.

Proof: From Theorem II.4 we have $\mathcal{R}(\mathcal{D})\subset\mathcal{U}$. We can now apply Theorem 3 to obtain $\sigma_{\mathcal{R}}(R)\subset M$, where $R = \mathcal{R}(\mathcal{D})$. Since $\mathcal{D}\subset R$ we have $\sigma_{\mathcal{R}}(\mathcal{D})\subset\sigma_{\mathcal{R}}(R)$, which proves the result.

Corollary 10. Let $P\subset\mathcal{D}\subset\mathcal{U}\subset M\subset P(\Omega)$ with $M\in\Sigma_{CIU}\cap\Sigma_{CDI}$, $\mathcal{U}\in\Sigma_{FDU}$ and $\mathcal{D}\in\Sigma_{PD}$. Suppose that $A\cap B\in S$ whenever $A,B\in P$, where $S = S(\Sigma_{PD},P)$. Then $\sigma_{\mathcal{R}}(P)\subset M$.

Proof: We apply Corollary II.5 to get $\mathcal{R}(P)\subset\mathcal{U}$. Then it follows from Theorem 3 that $\sigma_{\mathcal{R}}(R)\subset\mathcal{U}$, where $R = \mathcal{R}(P)$. The desired result is now easily obtained.

Corollary 11. Let $P\subset\mathcal{U}\subset M\subset P(\Omega)$ with $M\in\Sigma_{CIU}\cap\Sigma_{CDI}$ and $\mathcal{U}\in\Sigma_{FDU}\cap\Sigma_{PD}$. Suppose that $A\cap B\in S$ whenever $A,B\in P$, where $S = S(\Sigma_{FDU}\cap\Sigma_{PD},P)$. Then $\sigma_{\mathcal{R}}(P)\subset M$.

Proof: Theorem II.6 yields $\mathcal{R}(P)\subset\mathcal{U}$. If we now apply Theorem 3, it follows that $\sigma_{\mathcal{R}}(R)\subset M$, where $R = \mathcal{R}(P)$ and the result is proved.

Corollary 12. Let $P\subset\mathcal{D}\subset N\subset P(\Omega)$ with $N\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and $\mathcal{D}\in\Sigma_{PD}$. Suppose that $A\cap B\in S$ whenever $A,B\in P$, where $S = S(\Sigma_{PD},P)$. Then $\sigma_{\mathcal{R}}(P)\subset N$.

Proof: First apply Theorem II.2 to get $S\in\Sigma_D$. Next, note that by Theorem 4, we have $\sigma_{\mathcal{R}}(S)\subset N$. Since $P\subset S$, the proof is finished.

Corollary 13. Let $\mathcal{P}\subset\mathcal{U}\subset\mathcal{N}\subset\mathcal{P}(\Omega)$ with $\mathcal{N}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and $\mathcal{U}\in\Sigma_{FDU}\cap\Sigma_{PD}$. Suppose that $A\cap B\in\mathcal{S}$ whenever $A,B\in\mathcal{P}$, where $\mathcal{S} = S(\Sigma_{FDU}\cap\Sigma_{PD},\mathcal{P})$. Then $\sigma_{\mathcal{R}}(\mathcal{P})\subset\mathcal{N}$.

Proof: From Theorem II.6 we have $\mathcal{R}(\mathcal{P})\subset\mathcal{U}$. Thus on applying Theorem 4, we get $\sigma_{\mathcal{R}}(\mathcal{R})\subset\mathcal{N}$, where $\mathcal{R} = \mathcal{R}(\mathcal{P})$. This completes the proof.

Corollary 14. Let $\mathcal{U}\subset\mathcal{D}\subset\mathcal{K}\subset\mathcal{P}(\Omega)$ with $\mathcal{K}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ or $\mathcal{K}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $\mathcal{D}\in\Sigma_{PD}$. Suppose that $A\cup B\in\mathcal{S}$ whenever $A,B\in\mathcal{U}$, where $\mathcal{S} = S(\Sigma_{PD},\mathcal{U})$. Then $\sigma_{\mathcal{R}}(\mathcal{U})\subset\mathcal{K}$.

Proof: From Theorem II.9 we have $\mathcal{R}(\mathcal{U})\subset\mathcal{D}$. Now applying Theorem 3 or 4, we obtain $\sigma_{\mathcal{R}}(\mathcal{R})\subset\mathcal{D}$ where $\mathcal{R} = \mathcal{R}(\mathcal{U})$. The result is now immediate.

Corollary 15. Let $\mathcal{D}\subset\mathcal{U}\subset\mathcal{K}\subset\mathcal{P}(\Omega)$ with $\mathcal{K}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ or $\mathcal{K}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $\mathcal{U}\in\Sigma_{FI}\cap\Sigma_{FU}$. Suppose that $A-B\in\mathcal{S}$ whenever $A,B\in\mathcal{D}$, where $\mathcal{S} = S(\Sigma_{FI}\cap\Sigma_{FU},\mathcal{D})$. Then $\sigma_{\mathcal{R}}(\mathcal{D})\subset\mathcal{K}$.

Proof: First apply Theorem II.8 to get $\mathcal{R}(\mathcal{D})\subset\mathcal{U}$. Then from Theorem 3 or 4 it follows that $\sigma_{\mathcal{R}}(\mathcal{R})\subset\mathcal{K}$, where $\mathcal{R} = \mathcal{R}(\mathcal{D})$. Thus $\sigma_{\mathcal{R}}(\mathcal{D})\subset\mathcal{K}$.

Corollary 16. Let $\mathcal{T}\subset\mathcal{E}\subset\mathcal{K}\subset\mathcal{P}(\Omega)$ with $\mathcal{K}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ or $\mathcal{K}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $\mathcal{E}\in\Sigma_{SD}$. Let $\mathcal{S} = S(\Sigma_{SD},\mathcal{T})$ and suppose that one of the following holds: Either $A\cap B\in\mathcal{S}$ whenever

$A,B\varepsilon T$, or $A\cup B\varepsilon S$ whenever $A,B\varepsilon T$, or $A-B\varepsilon S$ whenever $A,B\varepsilon T$. Then $\sigma_{\mathcal{R}}(T)\subset K$.

Proof: Apply Theorem II.10 or II.11 or II.12, as is appropriate, to get $\mathcal{R}(T)\subset E$. Then apply Theorem 3 or 4 to get $\sigma_{\mathcal{R}}(R)\subset K$, where $R = \mathcal{R}(T)$. The desired result now follows.

Corollary 17. Let $E\subset P\subset K\subset P(\Omega)$ with $K\varepsilon\Sigma_{CIU}\cap\Sigma_{CDI}$ or $K\varepsilon\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $P\varepsilon\Sigma_{FI}$. Suppose that $A\cap B\varepsilon S$ whenever $A,B\varepsilon E$, where $S = S(\Sigma_{FI},E)$. Then $\sigma_{\mathcal{R}}(E)\subset K$.

Proof: From Theorem II.13 we have $\mathcal{R}(E)\subset P$. If we now apply Theorem 3 or 4, then $\sigma_{\mathcal{R}}(R)\subset K$, where $R = \mathcal{R}(E)$. Thus $\sigma_{\mathcal{R}}(E)\subset K$.

The foregoing results were all concerned with the σ-ring generated by a class of subsets. We now present some versions of these results for the generated σ-field.

Here is an extension of the usual monotone class theorem for σ-fields.

Theorem 18. Let $C\subset M\subset P(\Omega)$ with $M\varepsilon\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{\Omega}$. Suppose that whenever $A,B\varepsilon C$ we have $A\cap B\varepsilon S$ and $A^{c}\varepsilon S'$, where $S = S(\Sigma_{CIU}\cap\Sigma_{CDI}\cap\Sigma_{\Omega},C)$ and $S' = S(\Sigma_{CIU}\cap\Sigma_{CI}\cap\Sigma_{\Omega},C)$. Then $\sigma_{\mathfrak{f}}(C)\subset M$.

Proof: First, apply Theorem I.2 as follows. Define R on $P(\Omega)$ by ARB if and only $A\cap B\varepsilon S$. Taking $V = S$, we see from

Theorem 1(a,b) and Theorem II.1(a) that $G_I(C), G_I(S) \varepsilon \Sigma_{CIU} \cap \Sigma_{CDI} \cap \Sigma_\Omega$. Also I.2.1 is satisfied and, therefore, we can apply Theorem I.2 to get $S \varepsilon \Sigma_{FI}$. Next, apply Theorem I.1 again, with θ defined on $P(\Omega)$ by $\theta(A) = A^C$. Since $S = S'$, I.1.1 holds. Now if $A \varepsilon C$, then $\Phi = A \cap A^C$ and so $\Phi \varepsilon S$, since $S \varepsilon \Sigma_{FI}$. By Theorem 1(a,b) and Theorem II(a), we have $G_C \varepsilon \Sigma_{CIU} \cap \Sigma_{CDI} \cap \Sigma_\Omega$ and so, from Theorem I.1, it follows that $S \varepsilon \Sigma_C$. It is now easily verified that S is a σ-field and so $\sigma_\delta(C) \subset M$.

Remark. In the above result the operations of intersection and complement are not "interchangeable" (see remark following Theorem 3).

The next result extends the "σ-field version" of the theorem on normal classes mentioned earlier.

Theorem 19. Let $C \subset N \subset P(\Omega)$ with $N \varepsilon \Sigma_{CDU} \cap \Sigma_{CDI} \cap \Sigma_\Omega$. Suppose that whenever $A, B \varepsilon C$ we have $A \cap B \varepsilon S$ and $A^C \varepsilon S'$, where $S = S(\Sigma_{CDU} \cap \Sigma_{CDI}, C)$ and $S' = S(\Sigma_{CDU} \cap \Sigma_{CI}, C)$. Then $\sigma_\delta(C) \subset N$.

Proof: We proceed as in the previous theorem. Let R be defined on $P(\Omega)$ by ARB if and only if $A \cap B \varepsilon S$. Take $V = S$. Then we can apply Theorem 1(b,c) to get $G_I(C), G_I(S) \varepsilon \Sigma_{CDU} \cap \Sigma_{CDI}$. Now I.2.1 holds by hypothesis and from Theorem I.2 we have $S \varepsilon \Sigma_{FI}$. Define θ on $P(\Omega)$ by $\theta(A) = A^C$. In order to apply Theorem I.1, we first note that I.1.1 holds, since $S = S'$. Now, with $V = S$, from

Theorem 1(c,e) it follows that $G_C \cap S \in \Sigma_{CDU} \cap \Sigma_{CDI}$, since $S \in \Sigma_{CDU} \cap \Sigma_{CI}$ and Σ_{CDU} is closed under finite intersections. Applying Theorem I.1, we obtain $S \in \Sigma_C$. It now follows that S is a σ-field and thus $\sigma_{\delta}(C) \subset N$.

In the usual $\pi-\lambda$ theorem for σ-fields, it is possible to relax, somewhat, the condition that P be closed under finite intersections. The following result demonstrates this fact.

Theorem 20. Let $P \subset L \subset P(\Omega)$ with $L \in \Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$. Suppose that $A \cap B \in S$ whenever $A, B \in P$, where $S = S(\Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}, P)$. Then $\sigma_{\delta}(P) \subset L$.

Proof: Let R be defined on $P(\Omega)$ by ARB if and only if $A \cap B \in S$. Taking $V = S$, it follows from Theorem 1(a) and Theorem II.1(a,e) that $G_I(P), G_I(S) \in \Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$. Now I.2.1 holds by hypothesis and so by Theorem I.2, we have $S \in \Sigma_{FI}$. It is now clear that S is a σ-field and this completes the proof.

Theorem 21. Let $U \subset L \subset P(\Omega)$ with $L \in \Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$. Suppose that $A \cup B \in S$ whenever $A, B \in U$, where $S = S(\Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}, U)$. Then $\sigma_{\delta}(U) \subset L$.

Proof: Let R be defined on $P(\Omega)$ by ARB if and only if $A \cup B \in S$. Taking $V = S$, we have $G_U(U) \cap S, G_U(S) \cap S \in \Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$ from Theorem 1(a) and Theorem II.1(a,e), since $\Sigma_{CIU} \cap \Sigma_{\Omega}$ is closed

under finite intersections. Now I.2.1 holds by hypothesis and thus we can apply Theorem I.2 to get $S\varepsilon\Sigma_{FU}$. It follows that S is a σ-field and the proof is finished.

Theorem 22. Let $\mathcal{D}\subset M\subset P(\Omega)$ with $M\varepsilon\Sigma_{CIU}\cap\Sigma_{SD}\cap\Sigma_{\Omega}$. Suppose that $A-B\varepsilon S$ whenever $A,B\varepsilon\mathcal{D}$, where $S = S(\Sigma_{CIU}\cap\Sigma_{SD}\cap\Sigma_{\Omega},\mathcal{D})$. Then $\sigma_{\delta}(\mathcal{D})\subset M$.

Proof: Define R on $P(\Omega)$ by ARB if and only if $A-B\varepsilon S$. First, note that $S\varepsilon\Sigma_{C}$ since $\Sigma_{SD}\subset\Sigma_{PD}$ and $\Omega\varepsilon S$. Hence we also have $\Phi\varepsilon S$. On taking $V = S$, from Theorem 1(a,b) and Theorem II.1(a,f), it follows that $G_{D}(\mathcal{D}),H_{D}(S)\,\varepsilon\Sigma_{CIU}\cap\Sigma_{SD}\cap\Sigma_{\Omega}$, since $S\varepsilon\Sigma_{CIU}\cap\Sigma_{SD}$ and $\Sigma_{CIU}\cap\Sigma_{SD}\subset\Sigma_{CDI}$. Now I.2.1 holds by hypothesis and so from Theorem I.2 we have $S\varepsilon\Sigma_{D}$. Thus S is a σ-field and so $\sigma_{\delta}(\mathcal{D})\subset M$.

Theorem 23. Let $T\subset M\subset P(\Omega)$ with $M\varepsilon\Sigma_{CIU}\cap\Sigma_{SD}\cap\Sigma_{\Omega}$. Let $S = S(\Sigma_{CIU}\cap\Sigma_{SD}\cap\Sigma_{\Omega},T)$ and suppose that one of the following holds: Either $A\cap B\varepsilon S$ whenever $A,B\varepsilon T$, or $A\cup B\varepsilon S$ whenever $A,B\varepsilon T$. Then $\sigma_{\delta}(T)\subset M$.

Proof: Define R on $P(\Omega)$ by ARB if and only if $A\cap B\varepsilon S$ or ARB if and only if $A\cup B\,\varepsilon S$, as is appropriate. Taking $V = S$, we see from Theorem 1(a) and Theorem II.1(a,f) that $G_{I}(T),G_{I}(S)$ or $G_{U}(T),G_{U}(S)\,\varepsilon\Sigma_{CIU}\cap\Sigma_{SD}\cap\Sigma_{\Omega}$. Now I.2.1 holds by hypothesis and so, from Theorem I.2, we have $S\varepsilon\Sigma_{FI}$ or $S\varepsilon\Sigma_{FU}$. It follows that S is a σ-field. Consequently, $\sigma_{\delta}(T)\subset M$.

Remarks (1) Since $\Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega} = \Sigma_{CDU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$, equivalent forms of Theorems 20 and 21 are obtained by replacing $\Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$ by $\Sigma_{CDU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}$ and $S(\Sigma_{CIU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}, \mathcal{T})$ by $S(\Sigma_{CDU} \cap \Sigma_{PD} \cap \Sigma_{\Omega}, \mathcal{T})$, where $\mathcal{T} = \mathcal{C}$ or $\mathcal{P}$. Likewise, $\Sigma_{CIU} \cap \Sigma_{SD} \cap \Sigma_{\Omega} = \Sigma_{CDU} \cap \Sigma_{SD} \cap \Sigma_{\Omega}$ and so equivalent forms of Theorems 22 and 23 can be obtained by making the appropriate replacements.

(2) Theorem 23 does not directly follow from Theorems 20 and 21.

Theorem 24. Let $\mathcal{C} \subset \mathcal{U} \subset \mathcal{P}(\Omega)$ with $\mathcal{U} \in \Sigma_{CU} \cap \Sigma_{CI}$. Suppose that $A^c \in \mathcal{S}$ whenever $A \in \mathcal{C}$, where $\mathcal{S} = S(\Sigma_{CU} \cap \Sigma_{CI}, \mathcal{C})$. Then $\sigma_{\mathcal{J}}(\mathcal{C}) \subset \mathcal{U}$.

Proof: Define θ on $\mathcal{P}(\Omega)$ by $\theta(A) = A^c$. With $\mathcal{V} = \mathcal{S}$, from Theorem 1(d,e) we see that $\mathcal{G}_C \in \Sigma_{CU} \cap \Sigma_{CI}$. Now I.1.1 holds and so from Theorem I.1 we have $\mathcal{S} \in \Sigma_C$. Thus $\mathcal{S}$ is a σ-field and the result follows.

We end this section with several variants of the monotone class theorem, all of which are proved using Theorem 18 or 19 and the results from section II.

For σ-fields, the usual monotone class theorem and the π-λ theorem are consequences of the next result.

Corollary 25. Let $\mathcal{P} \subset \mathcal{D} \subset \mathcal{K} \subset \mathcal{P}(\Omega)$ with $\mathcal{K} \in \Sigma_{CIU} \cap \Sigma_{CDI}$ or $\mathcal{K} \in \Sigma_{CDU} \cap \Sigma_{CDI}$ and let $\mathcal{D} \in \Sigma_{PD} \cap \Sigma_{\Omega}$. Suppose that $A \cap B \in \mathcal{S}$ whenever $A, B \in \mathcal{P}$, where $\mathcal{S} = S(\Sigma_{PD} \cap \Sigma_{\Omega}, \mathcal{P})$. Then $\sigma_{\mathcal{J}}(\mathcal{P}) \subset \mathcal{K}$.

Proof: From Theorem II.15 we have $\delta(\mathcal{P})\subset\mathcal{D}$. Now apply Theorem 18 or 19 to get $\sigma_\delta(\mathcal{F})\subset\mathcal{K}$, where $\mathcal{F} = \delta(\mathcal{P})$. Since $\mathcal{P}\subset\mathcal{F}$, the result follows.

Corollary 26. Let $\mathcal{U}\subset\mathcal{D}\subset\mathcal{K}\subset\mathcal{P}(\Omega)$ with $\mathcal{K}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ or $\mathcal{K}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $\mathcal{D}\in\Sigma_{PD}\cap\Sigma_{\Omega}$. Suppose that $A\cup B\in\mathcal{S}$ whenever $A,B\in\mathcal{U}$, where $\mathcal{S} = S(\Sigma_{PD}\cap\Sigma_{\Omega},\mathcal{U})$. Then $\sigma_\delta(\mathcal{U})\subset\mathcal{K}$.

Proof: Theorem II.17 shows that $\delta(\mathcal{U})\subset\mathcal{D}$. By Theorem 18 or 19 we have $\sigma_\delta(\mathcal{F})\subset\mathcal{K}$, where $\mathcal{F} = \delta(\mathcal{U})$. Since $\mathcal{U}\subset\mathcal{F}$, the proof is complete.

Corollary 27. Let $\mathcal{C}\subset\mathcal{U}\subset\mathcal{K}\subset\mathcal{P}(\Omega)$ with $\mathcal{K}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ or $\mathcal{K}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $\mathcal{U}\in\Sigma_{FI}\cap\Sigma_{FU}$. Suppose that $A^c\in\mathcal{S}$ whenever $A\in\mathcal{C}$, where $\mathcal{S} = S(\Sigma_{FI}\cap\Sigma_{FU},\mathcal{C})$. Then $\sigma_\delta(\mathcal{C})\subset\mathcal{K}$.

Proof: First, apply Theorem II.16 to get $\delta(\mathcal{C})\subset\mathcal{U}$. Then from Theorem 18 or 19 it follows that $\sigma_\delta(\mathcal{F})\subset\mathcal{K}$, where $\mathcal{F} = \delta(\mathcal{C})$. Now $\mathcal{C}\subset\mathcal{F}$ and thus we have the desired result.

Corollary 28. Let $\mathcal{T}\subset\mathcal{E}\subset\mathcal{K}\subset\mathcal{P}(\Omega)$ with $\mathcal{K}\in\Sigma_{CIU}\cap\Sigma_{CDI}$ or $\mathcal{K}\in\Sigma_{CDU}\cap\Sigma_{CDI}$ and let $\mathcal{E}\in\Sigma_{SD}\cap\Sigma_{\Omega}$. Let $\mathcal{S} = S(\Sigma_{SD}\cap\Sigma_{\Omega},\mathcal{T})$ and suppose that one of the following holds: Either $A\cap B\in\mathcal{S}$ whenever $A,B\in\mathcal{T}$, or $A\cup B\in\mathcal{S}$ whenever $A,B\in\mathcal{T}$, or $A-B\in\mathcal{S}$ whenever $A,B\in\mathcal{T}$. Then $\sigma_\delta(\mathcal{T})\subset\mathcal{K}$.

Proof: Apply Theorem II.18, II.19 or II.20, as is appropriate, to get $\delta(\mathcal{T})\subset\mathcal{E}$. Then by Theorem 18 or 19 we

have $\sigma_{\delta}(F) \subset K$, where $F = \delta(T)$. Since $T \subset F$ it follows that $\sigma_{\delta}(T) \subset K$.

Corollary 29. Let $E \subset P \subset K \subset P(\Omega)$ with $K \in \Sigma_{\mathrm{CIU}} \cap \Sigma_{\mathrm{CDI}}$ or $K \in \Sigma_{\mathrm{CDU}} \cap \Sigma_{\mathrm{CDI}}$ and let $P \in \Sigma_{\mathrm{FI}}$. Suppose that $A \Delta B \in S$ whenever $A, B \in E$ and $S \in \Sigma_{\Omega}$, where $S = S(\Sigma_{\mathrm{FI}}, E)$. Then $\sigma_{\delta}(E) \subset K$.

Proof: From Theorem II.21, we have $\delta(E) \subset D$. Now apply Theorem 18 or 19 to get $\sigma_{\delta}(F) \subset K$, where $F = \delta(E)$. To finish the proof, we observe that $E \subset F$.

SECTION IV
FUNCTIONAL VERSIONS OF MINIMAL CLASS THEOREMS FOR σ-FIELDS

A functional version of the monotone class theorem and variants of this functional version are proved, using Zorn's Lemma, in Dellacherie and Meyer [4, p. 14-15, Theorem 21, 22.2,22.3]. In this section we extend their results and present other variants. The proofs are primarily based on the results in Section I and thus follow the methods given in the previous sections. Consequently, Zorn's Lemma is not required. As an aside, we obtain an extension of the functional version of the π-λ theorem. Our proof differs from the usual one (see Blumenthal and Getoor [2, p. 6, Theorem 2.3]), in that we do not use the π-λ theorem, but, instead, the results from Section I directly applied in a functional setting.

Throughout this section, Ω is a set; $A(\Omega)$ is the set of all real-valued functions on Ω; $B(\Omega)$ is the set of all bounded real-valued functions on Ω.

Let V be a subset of $B(\Omega)$. For each subset W of $B(\Omega)$ we define

$G_P(W) = \{g \varepsilon B(\Omega): fg \varepsilon V$ for every $f \varepsilon W\}$;

$G_S(W) = \{g \varepsilon B(\Omega): f+g \varepsilon U$ for every $f \varepsilon W\}$;

if a is real, then $G_{SM}(a) = \{g \varepsilon B(\Omega): ag \varepsilon V\}$;

$G_+ = \{g \varepsilon B(\Omega):\ g^+ \varepsilon V\}$.

Moreover, Σ_S is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that $\mathcal{U}$ is closed under addition;

Σ_{SM} is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that $\mathcal{U}$ is closed under scalar multiplication (scalars are always assumed to be real);

Σ_{DP} is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that $\mathcal{U}$ is closed under differences of non-negative functions;

Σ_P is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that $\mathcal{U}$ is closed under products;

Σ_+ is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that $\mathcal{U}$ is closed under the operation $f\to f^+$;

Σ_{LS} is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that if $f_n = \Sigma_{k=1}^{m_n} a_{kn} I_{A_{kn}}$, where $a_{kn}\geq 0, A_{kn}\subset\Omega, I_{A_{kn}}\varepsilon\mathcal{U}$ and f_n are uniformly bounded with $f_n\uparrow f$, then $f\varepsilon\mathcal{U}$;

Σ_{IP} is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that if $f_n\varepsilon\mathcal{U}$, where $f_n\geq 0$ and f_n are uniformly bounded with $f_n\uparrow f$ then $f\varepsilon\mathcal{U}$;

Σ_{MC} is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that if $f_n\varepsilon\mathcal{U}$, where f_n are uniformly bounded with $f_n\uparrow\downarrow f$, then $f\varepsilon\mathcal{U}$;

Σ_{UC} is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that if $f_n\varepsilon\mathcal{U}$, where f_n are uniformly bounded with $f_n\to f$ uniformly on Ω, then $f\varepsilon\mathcal{U}$;

Σ_1 is the class of all $\mathcal{U}\subset\mathcal{B}(\Omega)$ such that $1\varepsilon\mathcal{U}$.

Finally, $\mathcal{I}$ denotes the set $\{I_A\varepsilon\mathcal{B}(\Omega): A\subset\Omega\}$.

Recall that if $\mathcal{S}$ is a σ-field of subsets of Ω, then a real-valued function f on Ω is said to be $\mathcal{S}$-measurable if and only if $f^{-1}(B)\varepsilon\mathcal{S}$ for every Borel set B.

<u>Theorem 1</u>. Let $\mathcal{V}\subset\mathcal{B}(\Omega)$ such that $\mathcal{V}\in\Sigma_{LS}\cap\Sigma_{DP}\cap\Sigma_1$ and $\mathcal{V}\cap\mathcal{I}\in\Sigma_P$. Then $\mathcal{S}=\{A\subset\Omega : I_A\in\mathcal{V}\}$ is a σ-field and $\{f\in\mathcal{B}(\Omega): f$ is $\mathcal{S}$-measurable$\}\subset\mathcal{V}$.

<u>Proof</u>: Since $\mathcal{V}\in\Sigma_1$ and $1 = I_\Omega$, we have $\Omega\in\mathcal{S}$. Also $I_{A\cap B} = I_A I_B$ if $A,B\subset\Omega$ and thus the assumption $\mathcal{V}\cap\mathcal{I}\in\Sigma_P$ shows that $\mathcal{S}$ is closed under finite intersections. Next, since $I_{A-B} = I_A - I_B$ if $B\subset A\subset\Omega$ and $\mathcal{V}\in\Sigma_{DP}$ it follows that $\mathcal{S}$ is closed under proper differences and so $\mathcal{S}$ is a field. To show that $\mathcal{S}$ is a σ-field we let $A_n\uparrow A, A\in\mathcal{S}$. Then $I_{A_n}\uparrow I_A$ and $I_{A_n}\in\mathcal{V}$. Since $\mathcal{V}\in\Sigma_{LS}$, we see that $I_A\in\mathcal{V}$ and hence $A\in\mathcal{S}$. Thus $\mathcal{S}$ is a σ-field. To prove the second part, let $f\in\mathcal{B}(\Omega)$ be $\mathcal{S}$-measurable. We can write $f = f^+ - f^-$ where $f^+, f^-\geq 0$. Also $f^+, f^-\in\mathcal{B}(\Omega)$ and are $\mathcal{S}$-measurable.

In view of the fact that $\mathcal{V}\in\Sigma_{DP}$, it suffices to show that if $f\in\mathcal{B}(\Omega)$, $f\geq 0$ and f is $\mathcal{S}$-measurable, then $f\in\mathcal{V}$. But, in this case, we know there exist simple functions $f_n\uparrow f$, where $f_n = \Sigma_{k=1}^{m_n} a_{kn} I_{A_{kn}}$, $a_{kn}\geq 0$, $I_{A_{kn}}\in\mathcal{V}$ $(A_{kn}\in\mathcal{S})$ and f_n are uniformly bounded. Therefore, $f\in\mathcal{U}$ since $\mathcal{V}\in\Sigma_{LS}$ and the proof is finished.

<u>Theorem 2</u>. Let $\mathcal{V},\mathcal{W}\subset\mathcal{B}(\Omega)$.

(a) If $\mathcal{W}\subset\mathcal{V}$, then $1\in G_P(\mathcal{W})$.

(b) If $\mathcal{V}\in\Sigma_S$, then $G_P(\mathcal{W})\in\Sigma_S$.

(c) If $\mathcal{V}\in\Sigma_{SM}$, then $G_P(\mathcal{W})\in\Sigma_{SM}$.

(d) If $\mathcal{V}\in\Sigma_{UC}$ and a is real, then $G_{SM}(a)$, $G_S(\mathcal{W})$, $G_P(\mathcal{W})\in\Sigma_{UC}$.

(e) If $V\varepsilon\Sigma_{MC}$ and a is real, then $G_{SM}(a)$, $G_S(W)$, $G_+\varepsilon\Sigma_{MC}$.

(f) If $V\varepsilon\Sigma_{LS}$ and $W\subset I$, then $G_P(W)\varepsilon\Sigma_{LS}$.

(g) If $V\varepsilon\Sigma_{DP}$ and $W\subset I$, then $G_P(W)\varepsilon\Sigma_{DP}$.

Proof (a) If $W\subset V$, then for every $f\varepsilon W$, $f1 = f\varepsilon V$ so $1\varepsilon G_P(W)$.
(b) Let $g,h\varepsilon G_P(W)$. Then if $f\varepsilon W$, we have $fg,fh\varepsilon V$. Since $V\varepsilon\Sigma_S$ and $f(g+h) = fg+fh$, it follows that $g+h\varepsilon G_P(W)$.
(c) Let $g\varepsilon G_P(W)$. If a is real and $f\varepsilon W$, then $f(ag) = a(fg)$ and $fg\varepsilon V$. Hence $ag\varepsilon G_P(W)$ since $V\varepsilon\Sigma_{SM}$.
(d) Let a be real, $g_n\varepsilon G_{SM}(a)$ and suppose that g_n are uniformly bounded with $g_n\to g$ uniformly on Ω. Now ag_n are uniformly bounded, $ag_n\varepsilon V$ for each n and $ag_n\to ag$ uniformly on Ω. Since $V\varepsilon\Sigma_{UC}$, it follows that $ag\varepsilon V$ and so $g\varepsilon G_{SM}(a)$.
Let $g_n\varepsilon G_S(W)$, $f\varepsilon W$ and suppose that g_n are uniformly bounded with $g_n\to g$ uniformly on Ω. Then $f+g_n\varepsilon V$, $f+g_n$ are uniformly bounded and $f+g_n \to f+g$ uniformly on Ω. Since $V\varepsilon\Sigma_{UC}$, we have $g\varepsilon G_S(W)$.
Let $g_n\varepsilon G_P(W)$, $f\varepsilon W$ and suppose that g_n are uniformly bounded with $g_n\to g$ uniformly on Ω. Then $fg_n\varepsilon V$, fg_n are uniformly bounded and $fg_n\to fg$ uniformly on Ω. Now $V\varepsilon\Sigma_{UC}$ and thus $g\varepsilon G_P(W)$.
(e) Let a be real, $g_n\varepsilon G_{SM}(a)$, $g_n\uparrow\downarrow g$ and suppose that g_n are uniformly bounded. Then $ag_n\varepsilon V$. Now $ag_n\uparrow\downarrow ag$ if $a\geq 0$ and $ag_n\downarrow\uparrow ag$ if $a<0$. Moreover, ag_n are uniformly bounded. Since $V\varepsilon\Sigma_{MC}$, we have $g\varepsilon G_{SM}(a)$. Let $g_n\varepsilon G_S(W)$, $g_n\uparrow\downarrow g$ and suppose that g_n are uniformly bounded. If $f\varepsilon W$ then

$f+g_n$, $f+g_n\uparrow\downarrow f+g$ and $f+g_n$ are uniformly bounded, Since $V\varepsilon\Sigma_{MC}$, it follows that $g\varepsilon G_S(W)$.

Let $g_n \varepsilon G_+, g_n \uparrow\downarrow g$ and suppose that g_n are uniformly bounded. Then $g_n^+ \varepsilon V$, $g_n^+ \uparrow\downarrow g^+$ and g_n^+ are uniformly bounded. Since $V \varepsilon \Sigma_{MC}$ we get $g^+ \varepsilon V$ and so $g \varepsilon G_+$.

(f) Let $W \subset I$ and $f \varepsilon W$. Then $f = I_A$ for some $A \subset \Omega$. Let $g_n = \Sigma_{k=1}^{m_n} a_{kn} I_{A_{kn}}$, $a_{kn} \geq 0, A_{kn} \subset \Omega, I_{A_{kn}} \varepsilon G_P(W)$, with $g_n \uparrow g$ and suppose that g_n are uniformly bounded. Then $I_{A \cap A_{kn}} = I_A I_{A_{kn}} \varepsilon V$. Next, observe that the following identity holds: $I_A g_n = \Sigma_{k=1}^{m_n} a_{kn} I_A I_{A_{kn}} = \Sigma_{k=1}^{m_n} a_{kn} I_{A \cap A_{kn}}$. Moreover, $I_A g_n \uparrow I_A g$ and $I_A g_n$ are uniformly bounded. Since $V \varepsilon \Sigma_{LS}$, we see that $g \varepsilon G_P(W)$.

(g) Let $W \subset I$ and $f \varepsilon W$. Then $f = I_A$ for some $A \subset \Omega$. Let $g, h \varepsilon G_P(W)$ with $g, h \geq 0$. Then $I_A g, I_A h \varepsilon V$ and $I_A g, I_A h \geq 0$. Since $V \varepsilon \Sigma_{DP}$ and $I_A(g-h) = I_A g - I_A h$, it follows that $g - h \varepsilon G_P(W)$.

The following result extends the functional version of the $\pi-\lambda$ theorem.

Theorem 3. Let P be a class of subsets of Ω and $H \subset B(\Omega)$ with $C \subset H$ where $C = \{I_A : A \varepsilon P\}$ and $H \varepsilon \Sigma_{LS} \cap \Sigma_{DP} \cap \Sigma_1$. Suppose that $I_A I_B \varepsilon V$ whenever $I_A, I_B \varepsilon V$, where $V = S(\Sigma_{LS} \cap \Sigma_{DP} \cap \Sigma_1, C)$. Then we have $\{f \varepsilon B(\Omega): f$ is $\sigma_f(P)$- measurable$\} \subset H$.

Proof: Note that, by definition, $C \subset V$ and $V \varepsilon \Sigma_{LS} \cap \Sigma_{DP} \cap \Sigma_1$. Also $V \subset H$. Suppose we have shown that (*) if $A, B \subset \Omega$ and $I_A, I_B \varepsilon V$, then $I_A I_B \varepsilon V$. Then on applying Theorem 1, it follows that $S = \{A \subset \Omega : I_A \varepsilon V\}$ is a σ-field and $\{f \varepsilon B(\Omega): f$ is S-measurable$\} \subset V$. Since $C \subset V$, we have $P \subset S$ and so $\sigma_f(P) \subset S$. But $V \subset H$ and thus the result follows.

To complete the proof we only need to show that (*) holds. This is done by applying Theorem I.2. Here we use the version of the theorem as stated, with $I = \mathcal{I}$. Observe that $\mathcal{C} \cap \mathcal{I} = \mathcal{C}$ and so $\mathcal{V} = S(\Sigma_{LS} \cap \Sigma_{DP} \cap \Sigma_1, \mathcal{C}) = S(\Sigma_{LS} \cap \Sigma_{DP} \cap \Sigma_1, \mathcal{C} \cap \mathcal{I})$. Define R on $B(\Omega)$ by $f R g$ if and only if $fg \in \mathcal{V}$. Let I_A , $I_B \in \mathcal{C}$.

Then $A, B \in \mathcal{P}$ and by assumption we have I_A , $I_B \in \mathcal{V}$. Thus I.2.1 holds. Now by Theorem 2(a,f,g), it follows that $G_P(\mathcal{C}), G_P(\mathcal{V} \cap \mathcal{I}) \in \Sigma_{LS} \cap \Sigma_{DP} \cap \Sigma_1$ since $\mathcal{C} \subset \mathcal{V}, \mathcal{V} \cap \mathcal{I} \subset \mathcal{V}, \mathcal{C} \subset \mathcal{I}$ and $\mathcal{V} \cap \mathcal{I} \subset \mathcal{I}$. Apply Theorem I.2 to get $fg \in \mathcal{V}$ whenever $f \in \mathcal{V}, g \in \mathcal{V} \cap \mathcal{I}$. But then clearly (*) holds also and we are done.

<u>Remarks</u> (1) If $\mathcal{P}$ is closed under finite intersections, then it is easy to see that $I_A I_B = I_{A \cap B} \in \mathcal{V}$ whenever $A, B \in \mathcal{P}$ since $\mathcal{C} \subset \mathcal{V}$. Also if, for example, we suppose that given A, $B \in \mathcal{P}$ there exist $f_n = \Sigma_{k=1}^{m_n} a_{kn} I_{A_{kn}}$, $a_{kn} \geq 0$, $A_{kn} \in P$, (f_n) uniformly bounded such that $f_n \uparrow I_{A \cap B}$, or there exist I_C, I_D with C, $D \in \mathcal{P}$ such that $I_{A \cap B} = I_C - I_D$, then clearly $I_A I_B \in \mathcal{V}$.
(2) If, in the preceding result, we had $\mathcal{H} \subset A(\Omega)$ instead of $\mathcal{H} \subset B(\Omega)$, it is possible to prove a similar result which would show that $\{f \in A(\Omega) : f$ is $\sigma_f(\mathcal{P})$-measurable$\} \subset \mathcal{H}$, provided we suitably modify the definitions of Σ_{LS}, Σ_{DP} and Σ_1 and reprove the relevant portions of Theorems 1 and 2 that are affected by the changes.

If $\mathcal{V} \subset B(\Omega)$ is a vector space and $\mathcal{V} \in \Sigma_+$, then in view of the identities $f \wedge g = (f+g) - (f \vee g)$ and $(f \vee g) = (f-g)^+ + g$ it is clear that $\mathcal{V}$ is also closed under $\vee$ and $\wedge$. Moreover, if

$A,B\subset\Omega$ and $I_A, I_B \in V$, then $I_A\, I_B = I_A \wedge I_B \in V$. These facts will be used in the proof of the next result.

<u>Theorem 4</u>. Let $V\subset B(\Omega)$ be a vector space such that $V\in \Sigma_{LS}\cap\Sigma_+\cap\Sigma_1$. Suppose also that if $f_n \uparrow I_A, f_n \geq 0, f_n \in V$, $A\subset\Omega$, then $I_A \in V$. Then $S = \{A\subset\Omega : I_A \in V\}$ is a σ-field and $V = \{f\in B(\Omega) : f$ is S-measurable$\}$.

<u>Proof</u>: On account of Theorem 1, it is clear that S is a σ-field and moreover, $V\supset\{f\in B(\Omega) : f$ is S-measurable$\}$. Conversely, let $f\in V$. We can write $f = f^+ - f^-$, where $f^+, f^- \geq 0$ Also $f^+ \in V$ by hypothesis and hence $f^- \in V$. It thus suffices to show that if $f\in V, f\geq 0$, then f is S-measurable. So let $f\in V, f\geq 0$. If $a>0$, letting $h = \frac{1}{a} f$ and $f_n = 1 \wedge (n(h-(h\wedge 1)))$, we see that $f_n \in V$ since V is a vector space and $V\in\Sigma_+$. It can also be verified that $f_n \uparrow I_B$, where $B = f^{-1}(a,\infty)$. Since $f_n \geq 0$, we use the hypothesis again to obtain $I_B \in V$. Thus $B\in S$. Next, if $a<0$ then $B = \Omega\in S$ and if $a = 0$, then $B = f^{-1}(0,\infty) = f^{-1}(\cup_{n=1}^{\infty}(\frac{1}{n},\infty)) = \cup_{n=1}^{\infty} f^{-1}(\frac{1}{n},\infty)$ and so $B\in S$, since we already showed that $f^{-1}(a,\infty)\in S$ for $a>0$. We have now shown that f is S-measurable and the proof is complete.

If $C\subset B(\Omega)$, then $\sigma_f(C)$ will denote the intersection of all σ-fields with respect to which every function in C is measurable.

<u>Corollary 5</u>. Let $C\subset H\subset B(\Omega)$. Suppose that there exists a

vector space V with $C\subset V\subset H$ and such that $V\in\Sigma_{IP}\cap\Sigma_{+}\cap\Sigma_{1}$. Then $\{f\in B(\Omega)\colon f$ is $\sigma_f(C)$-measurable$\}\subset H$.

Proof: Let $S = \{A\subset\Omega\colon I_A\in V\}$. From Theorem 4, it follows that every $f\in C$ is S-measurable. Thus every $\sigma_f(C)$-measurable function in $B(\Omega)$ is also S-measurable. But now, using Theorem 4 again, we see that every $\sigma_f(C)$-measurable function in $B(\Omega)$ is in V and hence in H.

Here is an extension of a variant of the functional version of the monotone class theorem (see Dellacherie and Meyer [4, 22.3, p.15]).

Theorem 6. Let $C\subset H\subset B(\Omega)$ with $H\in\Sigma_{MC}$. Denote by V the set $S(\Sigma_{MC}, C)$. Suppose that $V\in\Sigma_1$ and also that whenever $f, g\in C$ and a is real, we have $f^+, af, f+g\in V$. Then $\{f\in B(\Omega)\colon f$ is $\sigma_f(C)$-measurable$\}\subset H$.

Proof: We will first show, using Theorem I.1 and I.2, that V is a vector space. Define R on $B(\Omega)$ by fRg if and only if $f+g\in V$. Then $G_S(C)$, $G_S(V)\in\Sigma_{MC}$ by Theorem 2(e). Now I.2.1 holds by hypothesis and so by Theorem I.2, we have $V\in\Sigma_S$. Next let a be real and define θ_a on $B(\Omega)$ by $\theta_a(g) = ag$. Note that I.1.1 holds by hypothesis. By Theorem 2(e), we have $G_{SM}(a)\in\Sigma_{MC}$ and so from Theorem I.1 it follows that $V\in\Sigma_{SM}$. Thus V is a vector space. Now apply Theorem I.1 again, as follows: Let θ be defined on $B(\Omega)$ by $\theta(g) = g^+$. Theorem 2(e) shows that $G_+\in\Sigma_{MC}$. Since I.1.1 holds by

hypothesis, we can apply Theorem I.1 to show that $V\varepsilon\Sigma_+$. It is easy to see that $C\subset V\subset H$ and thus from Corollary 5 the desired result follows.

Remark. If C is a vector space and C is closed under $\wedge$, then the identity $f\vee g = (f+g)-(f\wedge g)$ shows that C is closed under $\vee$. Hence $C\varepsilon\Sigma_+$. Thus, if C is a vector space with $1\varepsilon C$ and C is closed under $\wedge$, then C satisfies the hypothesis of Theorem 6. More generally, if there exist $f_n\varepsilon C$ such that f_n are uniformly bounded and $f_n\uparrow\downarrow f$, then $f\varepsilon V$. Therefore, taking f to be $1, f^+, af, f+g$ we obtain conditions to ensure that we will have $1, f^+, af, f+g\,\varepsilon V$ as the hypothesis requires. Similar remarks will apply to all results of this kind.

Theorem 7. Let $V\subset B(\Omega)$ be a vector space with $V\varepsilon\Sigma_{UC}\cap\Sigma_1$. Suppose, in addition, that for each $n = 1,2,\ldots$ V is closed under the operation $f\to f^n$. Then $V\varepsilon\Sigma_+$.

Proof: Let $f\varepsilon V$. Since f is bounded, there exists M such that $|f(w)|\leq M$ for every $w\varepsilon\Omega$. Let g be the function $x\to|x|$, x real. By the Weierstrass Theorem, there exists a sequence (p_n) of polynomials such that $p_n\to g$ uniformly on $[-M,M]$. So $p_n\circ f\to|f|$ uniformly on Ω. Let $p_n(x) = a_{on}+a_{1n}x+\ldots+a_{mn}x^m$. Then $(p_n\circ f)(w) = a_{on}+a_{1n}f(w)+\ldots+a_{mn}f^m(w)$ and thus $p_n\circ f = a_{on}+a_{1n}f+\ldots+a_{mn}f^m$. Since V is a vector space, $1\varepsilon V$

and V is closed under $f \to f^n$ it follows that $p_n \circ f \in V$. But $V \in \Sigma_{UC}$ and $p_n \circ f$ are uniformly bounded since f is bounded. Hence $|f| \in V$ and the identity $f^+ = \frac{1}{2}(f + |f|)$ shows that $f^+ \in V$ since V is a vector space.

Corollary 8. Let $C \subset H \subset B(\Omega)$ with $H \in \Sigma_{MC}$. Suppose that there exists a vector space V such that $C \subset V \subset H$, $V \in \Sigma_{UC} \cap \Sigma_1$ and V is closed under the operation $f \to f^n$ for $n = 1, 2, \ldots$ Then $\{f \in B(\Omega): f \text{ is } \sigma_f(C)\text{-measurable}\} \subset H$.

Proof: Let $V' = S(\Sigma_{MC}, V)$. Since V is a vector space and $V \subset V'$ we note if $f, g \in V$ and a is real, then $f+g \in V'$ and $af \in V'$. Also $1 \in V'$. In view of Theorem 7, it follows that if $f \in V$ then $f^+ \in V'$ since $V \subset V'$. Now apply Theorem 6 to obtain $\{f \in B(\Omega): f \text{ is } \sigma_f(V)\text{-measurable}\} \subset H$. To finish the proof, observe that $\sigma_f(C) \subset \sigma_f(V)$.

Theorem 9. Let $C \subset L \subset U \subset H \subset B(\Omega)$ with $H \in \Sigma_{MC}$, $U \in \Sigma_{UC}$ and let $V = S(\Sigma_{UC}, L)$. Suppose that $V \in \Sigma_1$ and also that whenever $f, g \in L$ and a is real, we have af, $f+g \in V$. Finally let $V' = S(\Sigma_{UC} \cap \Sigma_S \cap \Sigma_{SM} \cap \Sigma_1, C)$ and suppose that $fg \in V'$ whenever $f, g \in C$. Then $\{f \in B(\Omega): f \text{ is } \sigma_f(C)\text{-measurable}\} \subset H$.

Proof: First, we show, using Theorems I.1 and I.2, that V is a vector space. Define the relation R on $B(\Omega)$ by $f R g$ if and only if $f+g \in V$. From Theorem 2(d) we have $G_S(L)$, $G_S(V) \in \Sigma_{UC}$. Next note that I.2.1 holds by hypothesis. By

Theorem I.2, we have $V\varepsilon\Sigma_S$. Next, let a be real and define on $B(\Omega)$ by $\theta_a(g) = ag$. Now I.1.1 holds by hypothesis. Also by Theorem 2(d) we have $G_{SM}(a)\varepsilon\Sigma_{UC}$. Thus Theorem I.1 yields $V\varepsilon\Sigma_{SM}$. V is therefore a vector space. Since $V\varepsilon\Sigma_{UC}\cap\Sigma_1$ and $C\subset V$, it is clear that $V'\subset V$. Now apply Theorem I.2 again. Let R' be defined on $B(\Omega)$ by $fR'g$ if and only if $fg\varepsilon V'$. By Theorem 2(a,b,c,d,), it follows that

$G_P(C), G_P(V')\varepsilon\Sigma_{UC}\cap\Sigma_S\cap\Sigma_{SM}\cap\Sigma_1$. I.2.1 holds by hypothesis and so we can apply Theorem I.2 to get $V'\varepsilon\Sigma_P$. Thus V' is closed under the operation $f\to f^n$ for n = 1,2, . . . Observe also that $C\subset V'\subset H$ and that the other conditions needed in order to apply Corollary 8 are satisfied. Hence it follows that $\{f\varepsilon B(\Omega): f \text{ is } \sigma_f(C)\text{-measurable}\}\subset H$.

<u>Theorem 10</u>. Let $C\subset L\subset H\subset B(\Omega)$ with $H\varepsilon\Sigma_{MC}\cap\Sigma_{UC}$ and let $V = S(\Sigma_{MC}\cap\Sigma_{UC}, L)$. Suppose that $V\varepsilon\Sigma_1$ and also that whenever $f, g\varepsilon L$ and a is real, we have af, $f+g\varepsilon V$. Let $V' = S(\Sigma_{UC}\cap\Sigma_S\cap\Sigma_{SM}\cap\Sigma_1, C)$ and suppose that whenever $f, g\varepsilon C$ we have $fg\varepsilon V'$. Then $\{f\varepsilon B(\Omega): f \text{ is } \sigma_f(C)\text{-measurable}\}\subset H$.

<u>Proof</u>: As in the previous theorem, we first show that V is a vector space. Define R on $B(\Omega)$ by fRg if and only if $f+g\varepsilon V$. Note that I.2.1 holds by hypothesis. From Theorem 2(d,e) we get $G_S(L), G_S(V)\varepsilon\Sigma_{MC}\cap\Sigma_{UC}$. Thus Theorem I.2 can be applied and we have $V\varepsilon\Sigma_S$. Next, let a be real and define θ_a on $B(\Omega)$ by $\theta_a(g) = ag$. I.1.1 holds by hypothesis and from Theorem 2(d,e) we have $G_{SM}(a)\varepsilon\Sigma_{MC}\cap\Sigma_{UC}$. By Theorem I.1, it

follows that $V \in \Sigma_{SM}$ and thus V is a vector space. Hence it is clear that $V' \subset V$. Now apply Theorem I.2 again, as follows. Let R' be defined on $B(\Omega)$ by $f R' g$ if and only if $fg \in V'$. Then I.2.1 holds. On applying Theorem 2(a,b,c,d) we get $G_P(C), G_P(V') \in \Sigma_{UC} \cap \Sigma_S \cap \Sigma_{SM} \cap \Sigma_1$. Therefore we have $V' \in \Sigma_P$ from Theorem I.2. It follows that V' is closed under the operation $f \to f^n$ for n = 1,2, . . . Also $C \subset V' \subset H$ and it is easily verified that the remaining conditions in the hypothesis of Corollary 8 are satisfied. Consequently, we obtain $\{f \in B(\Omega) : f$ is $\sigma_f(C)$-measurable$\} \subset H$.

It is now quite easy in view of Theorem 10, to extend the functional version of the monotone class theorem and the remaining variant given in Dellacherie and Meyer [4, Theorem 21 and 22.2, p. 14, 15].

<u>Corollary 11</u>. Let $C \subset H \subset B(\Omega)$. Suppose that H is a vector space with $H \in \Sigma_{IP} \cap \Sigma_{UC} \cap \Sigma_1$. Let $V = S(\Sigma_{UC} \cap \Sigma_S \cap \Sigma_{SM} \cap \Sigma_1, C)$ and suppose that whenever $f, g \in C$ we have $fg \in V$. Then $\{f \in B(\Omega) : f$ is $\sigma_f(C)$-measurable$\} \subset H$.

<u>Proof</u>: We first show that, in fact, $H \in \Sigma_{MC}$. Let $f_n \uparrow\downarrow f, f_n \in H$ and assume that f_n are uniformly bounded. Then there exists K such that $|f_n(\omega)| \leq K$ for every $\omega \in \Omega$ and for n = 1,2, . . . So $f_n + K \geq 0$ and $K - f_n \geq 0$. Now $f_n + K, K - f_n \in H$ since H is a vector space and clearly $f_n + K$ and $K - f_n$ are uniformly bounded. Moreover, $f_n + K \uparrow f + K$ if $f_n \uparrow f$ and $K - f_n \uparrow K - f$ if

$f_n \downarrow f$. Since $H \in \Sigma_{IP}$ we have $f+K \in H$ if $f_n \uparrow f$, and $K-f \in H$ if $f_n \downarrow f$. In either case, it follows that $f \in H$, since H is a vector space. Thus $H \in \Sigma_{MC}$. Note that if $V' = S(\Sigma_{MC} \cap \Sigma_{UC}, H)$, then we have $V' \in \Sigma_1$. Also whenever $f, g \in H$ and a is real, af, $f+g \in H \subset V'$. The remaining conditions in the hypothesis of Theorem 10 clearly hold. Therefore, we obtain $\{f \in B(\Omega) : f$ is $\sigma_f(C)$-measurable$\} \subset H$.

Corollary 12. Let $C \subset H \subset B(\Omega)$ with $H \in \Sigma_{MC} \cap \Sigma_{UC}$. Let $V = S(\Sigma_{MC} \cap \Sigma_{UC}, C)$ and $V' = S(\Sigma_{UC} \cap \Sigma_S \cap \Sigma_{SM} \cap \Sigma_1, C)$. Suppose that $V \in \Sigma_1$ and also that whenever $f, g \in C$ and a is real, we have af, $f+g$ and $fg \in V'$. Then $\{f \in B(\Omega) : f$ is $\sigma_f(C)$-measurable$\} \subset H$.

Proof: All conditions needed in order to apply Theorem 10 are clearly satisfied and the result follows immediately.

Remark. In Theorem 6, the operations of taking the positive part, scalar multiplication and addition can be interchanged. Thus, for example, in addition to V we could have defined $V' = S(\Sigma_{MC} \cap \Sigma_S, C)$, $V'' = S(\Sigma_{MC} \cap \Sigma_S \cap \Sigma_{SM}, C)$ and altered the third sentence of the hypothesis to read as follows: Suppose that $V \in \Sigma_1$ and also that whenever $f, g \in C$ and a is real, we have $f+g \in V$, $af \in V'$ and $f^+ \in V''$.

Note that, in this case, the same proof will show $V \in \Sigma_S$, so that $V' = V$. Consequently, the rest of the proof yields

$V' = V \in \Sigma_{SM}$. It thus follows that $V'' = V' = V$ and hence the identical conclusion is obtained. Similar remarks apply to Theorems 9 and 10 with regard to the operations of scalar multiplication and addition. Thus the same results are obtained even if, for example, we modify the second sentence of each hypothesis to read:

Suppose that $V \in \Sigma_1$ and also that whenever $f, g \in L$ and a is real, we have $f+g \in V$, $af \in V''$. Here $V'' = S(\Sigma_{UC} \cap \Sigma_S, L)$ in Theorem 9 and $V'' = S(\Sigma_{MC} \cap \Sigma_{UC} \cap \Sigma_S, L)$ in Theorem 10.

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BIOGRAPHICAL SKETCH

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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